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The Turán number of disjoint copies of paths ☆

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ABSTRACT

The Turán number of a graph H, denoted by $ex(n, H)$, is the maximum number of edges in a simple graph of order n which does not contain H as a subgraph. In this paper, we determine the value $ex(n, k \cdot P_3)$ and characterize all extremal graphs for all positive integers n and k, where $k \cdot P_3$ is k disjoint copies of a path on three vertices. This extends a result of Bushaw and Kettle (2011), which solved the conjecture proposed by Gorgol (2011).

1. Introduction

Our notation in this paper is standard (see, e.g. [6]). Let $G = (V(G), E(G))$ be a simple graph, where $V(G)$ is the vertex set with n vertices and $E(G)$ is the edge set with size $e(G)$. The *degree* of $v \in V(G)$, the number of edges incident to v, is denoted by $d_G(v)$ and the set of neighbors of v is denoted by $N(v)$. If u and v in $V(G)$ are adjacent, we say that u *hits* v or v *hits* u. If u and v are not adjacent, we say that u *misses* v or v *misses* u. For $S \subseteq V(G)$, the induced subgraph of G by S is denoted by $G[S]$. Let G and H be two disjoint graphs, denote by $G \bigcup H$ the disjoint union of G and H and by $k \cdot G$ the disjoint union of k copies of a graph G. Denote by $G + H$ the graph obtained from $G \bigcup H$ by adding edges between all vertices of G and all vertices of H. Moreover, denote by P_l a path on l vertices and by M_t the disjoint union of $\lfloor \frac{t}{2} \rfloor$ disjoint copies of edges and $\lceil \frac{t}{2} \rceil - \lfloor \frac{t}{2} \rfloor$ isolated vertex (maybe no isolated vertex). We often refer to a path by the nature sequence of its vertices, writing, say, $P_l = x_1 x_2 \dots x_l$ and calling P_l a path from x_1 to x_l.

The Turán number of a graph H, denoted by $ex(n, H)$, is the maximum number of edges in a graph of order n which does not contain H as a subgraph. Denote by $G_{ex}(n, H)$ a graph on n vertices with $ex(n, H)$ edges containing no H as a subgraph and call this graph an *extremal graph* for H. Often, there are several extremal graphs. In 1941, Turán proved that the extremal graph without containing K_r as a subgraph is the Turán graph $T_{r-1}(n)$. Later, Moon [19] (only when $r - 1$ divides $n - k + 1$) and Simonovits [21] showed that $K_{k-1} + T_{r-1}(n - k + 1)$ is the unique extremal graph containing no $k \cdot K_r$ for sufficiently large n.

In 1959, Erdős and Gallai [7] proved the following well known result.

Theorem 1.1 ([7]). *If G is a simple graph with $n \geq k$ vertices, then $ex(n, P_k) \leq \frac{1}{2}(k-2)n$ with equality if and only if $n = (k-1)t$. Moreover the extremal graph is $\bigcup_{i=1}^{t} K_{k-1}$.*

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Recently, Gorgol [13] studied the Turán number of disjoint copies of any connected graphs. Let H be any connected graph on l vertices, with the aid of the two graphs $G_{ex}(n-kl+1,H)\bigcup K_{kl-1}$ and $G_{ex}(n-k+1,H)+K_{k-1}$, she presented a lower bound for $ex(n,k\cdot H)$. In particular, she proved the following.

Theorem 1.2 (*[13]*).

$$ex(n,2\cdot P_3) = \left\lfloor \frac{n-1}{2} \right\rfloor + n - 1, \qquad \text{for} \quad n \geq 9;$$
$$ex(n,3\cdot P_3) = \left\lfloor \frac{n}{2} \right\rfloor + 2n - 4, \qquad \text{for} \quad n \geq 14.$$

Furthermore, based on Theorem 1.2 and the lower bound of k disjoint copies of connected graph, she proposed the following conjecture.

Conjecture 1.3 (*[13]*).

$$ex(n,k\cdot P_3) = \left\lfloor \frac{n-k+1}{2} \right\rfloor + (k-1)n - \frac{k(k-1)}{2}$$

for n sufficiently large.

Bushaw and Kettle [3] proved Conjecture 1.3 and characterized all extremal graphs. Their result is as follows.

Theorem 1.4 (*[3]*).

$$ex(n,k\cdot P_3) = \binom{k-1}{2} + (n-k+1)(k-1) + \left\lfloor \frac{n-k+1}{2} \right\rfloor \qquad \text{for} \quad n \geq 7k.$$

Moreover, the only extremal graph is $K_{k-1}+M_{n-k+1}$.

In fact, Gorgol in [13] also conjecture that the lower bound is sharp for $k\cdot P_3$. Based on the proof of Conjecture 1.3, Bushaw and Kettle [3] conjectured that the extremal graph is unique for $n > 5k-1$. Their conjecture can be stated as follows.

Conjecture 1.5 (*[3,13]*).

$$ex(n,k\cdot P_3) = \begin{cases} \binom{3k-1}{2} + \left\lfloor \frac{n-3k+1}{2} \right\rfloor, & \text{for} \quad 3k \leq n \leq 5k-1; \\ \binom{k-1}{2} + (n-k+1)(k-1) + \left\lfloor \frac{n-k+1}{2} \right\rfloor, & \text{for} \quad n \geq 5k-1. \end{cases}$$

In [3], Bushaw and Kettle also determined the Turán number of k disjoint copies of P_l with $l \geq 4$ and also characterized all extremal graphs for sufficiently large n. The related results on the Turán number of paths, forests may be referred to [1,2,8,18] and the references therein. There are also many hypergraph Turán problems [11,10,17] of paths and cycles and some results of the disjoint union of hypergraphs [4,14]. For Turán numbers of graphs and hypergraphs, there are several excellent surveys [12,15,20] for more information.

There are very few cases when the Turán number $ex(n,F)$ is known exactly for all n. Erdős and Gallai [7] showed that

$$ex(n,M_{2k+2}) = \max\left\{ \binom{2k+1}{2}, \binom{k}{2} + k(n-k) \right\}.$$

$ex(n,P_k)$ was determined for all n and k by Faudree and Schelp [8] and independently by Kopylov [16]. Füredi and Gunderson [9] determined $ex(n,C_{2k+1})$ for all k and n and characterized all extremal graphs.

In this paper, we determine $ex(n,k\cdot P_3)$ and characterize all extremal graphs for all k and n, which confirms Conjecture 1.5. The main result in this paper is as follows.

Theorem 1.6.

$$ex(n,k\cdot P_3) = \begin{cases} \binom{n}{2}, & \text{for} \quad n < 3k; \\ \binom{3k-1}{2} + \left\lfloor \frac{n-3k+1}{2} \right\rfloor, & \text{for} \quad 3k \leq n < 5k-1; \\ \binom{3k-1}{2} + k, & \text{for} \quad n = 5k-1; \\ \binom{k-1}{2} + (n-k+1)(k-1) + \left\lfloor \frac{n-k+1}{2} \right\rfloor, & \text{for} \quad n > 5k-1. \end{cases}$$