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Eigenvalues of non-regular linear quasirandom hypergraphs

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ABSTRACT

Chung, Graham, and Wilson proved that a graph is quasirandom if and only if there is a large gap between its first and second largest eigenvalue. Recently, the authors extended this characterization to coregular k-uniform hypergraphs with loops. However, for $k \geq 3$ no k-uniform hypergraph is coregular.

In this paper we remove the coregular requirement. Consequently, the characterization can be applied to k-uniform hypergraphs; for example it is used in Lenz and Mubayi (2015) [5] to show that a construction of a k-uniform hypergraph sequence has some quasirandom properties. The specific statement that we prove here is that if a k-uniform hypergraph satisfies the correct count of a specially defined four-cycle, then its second largest eigenvalue is much smaller than its largest one.

1. Introduction

The authors [4] recently proved a hypergraph generalization of the famous Chung–Graham–Wilson [1] characterization of quasirandom graph sequences. However, the proof only applied to coregular hypergraph sequences and no k-uniform hypergraph is coregular for $k \geq 3$. In this paper we prove this equivalence for all k-uniform hypergraph sequences, not just the coregular ones. This paper should be viewed as a companion to [4] and many details and definitions that appear in [4] are not repeated here. This characterization has already been used in [5].

Definition 1. Let Ω be a set and k an integer. A k-multiset S on Ω is a function $S : \Omega \to \mathbb{Z}^{\geq 0}$ such that $\sum_{x \in \Omega} S(x) = k$. A k-uniform hypergraph with loops H consists of a vertex set $V(H)$ and an edge set $E(H)$ which is a collection of k-multisets on $V(H)$. A k-uniform hypergraph with loops is *coregular* if there is a positive integer d such that for every $(k-1)$-multiset S on $V(H)$,

$$|\{T \in E(H) : \forall x \in V(H), S(x) \leq T(x)\}| = d.$$

A k-uniform hypergraph is a k-uniform hypergraph with loops H such that for every $S \in E(H)$, $im(S) = \{0, 1\}$. A graph is a 2-uniform hypergraph.

Remarks.

• Informally, in a k-uniform hypergraph with loops every edge has size exactly k but a vertex is allowed to be repeated inside of an edge.

• For $k = 2$, a d-regular graph is a coregular 2-uniform hypergraph with loops, since each 1-multiset (i.e. a vertex) is contained in exactly d edges. But for $k \geq 3$, a k-uniform hypergraph cannot be coregular. For example, if H is a 3-uniform hypergraph then H is not coregular because for each vertex x, the multiset $\{x, x\}$ is not contained in any edge of H.

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Let $k \geq 2$ be an integer and let π be a proper partition of k, by which we mean that π is an unordered list of at least two positive integers whose sum is k. For the partition π of k given by $k = k_1 + \cdots + k_t$, we will abuse notation by saying that $\pi = k_1 + \cdots + k_t$. If F and G are k-uniform hypergraphs with loops, a *labeled copy of F in H* is an edge-preserving injection $V(F) \to V(H)$, i.e. an injection $\alpha : V(F) \to V(H)$ such that if E is an edge of F, then $\{\alpha(x) : x \in E\}$ is an edge of H. The following is our main theorem.

Theorem 2. *Let $0 < p < 1$ be a fixed constant and let $\mathcal{H} = \{H_n\}_{n\to\infty}$ be a sequence of k-uniform hypergraphs with loops such that $|V(H_n)| = n$ and $|E(H_n)| \geq p\binom{n}{k}$. Let $\pi = k_1 + \cdots + k_t$ be a proper partition of k and let $\ell \geq 1$. Assume that $\mathcal{H}$ satisfies the property*

- $\texttt{Cycle}_{4\ell}[\pi]$: *the number of labeled copies of $C_{\pi,4\ell}$ in H_n is at most $p^{|E(C_{\pi,4\ell})|} n^{|V(C_{\pi,4\ell})|} + o(n^{|V(C_{\pi,4\ell})|})$, where $C_{\pi,4\ell}$ is the hypergraph cycle of type π and length 4ℓ defined in [4, Section 2].*

Then $\mathcal{H}$ satisfies the property

- $\texttt{Eig}[\pi]$: *$\lambda_{1,\pi}(H_n) = pn^{k/2} + o(n^{k/2})$ and $\lambda_{2,\pi}(H_n) = o(n^{k/2})$, where $\lambda_{1,\pi}(H_n)$ and $\lambda_{2,\pi}(H_n)$ are the first and second largest eigenvalues of H_n with respect to π, defined in Section 2.*

When Theorem 2 is combined with [4, Section 1.2], we obtain the following theorem which generalizes many parts of [1] to hypergraphs.

Theorem 3. *Let $0 < p < 1$ be a fixed constant and let $\mathcal{H} = \{H_n\}_{n\to\infty}$ be a sequence of k-uniform hypergraphs with loops such that $|V(H_n)| = n$ and $|E(H_n)| \geq p\binom{n}{k} + o(n^k)$. Let $\pi = k_1 + \cdots + k_t$ be a proper partition of k. The following properties are equivalent:*

- $\texttt{Eig}[\pi]$: *$\lambda_{1,\pi}(H_n) = pn^{k/2} + o(n^{k/2})$ and $\lambda_{2,\pi}(H_n) = o(n^{k/2})$, where $\lambda_{1,\pi}(H_n)$ and $\lambda_{2,\pi}(H_n)$ are as defined in Section 2.*
- $\texttt{Expand}[\pi]$: *For all $S_i \subseteq \binom{V(H_n)}{k_i}$ where $1 \leq i \leq t$,*
$$e(S_1, \ldots, S_t) = p\prod_{i=1}^{t} |S_i| + o\left(n^k\right)$$
where $e(S_1, \ldots, S_t)$ is the number of tuples $(s_1, \ldots, s_t)$ such that $s_1 \cup \cdots \cup s_t$ is a hyperedge and $s_i \in S_i$.
- $\texttt{Count}[\pi\text{-linear}]$: *If F is an f-vertex, m-edge, k-uniform, π-linear hypergraph, then the number of labeled copies of F in H_n is $p^m n^f + o(n^f)$. The definition of π-linear appears in [4, Section 1.2].*
- $\texttt{Cycle}_4[\pi]$: *The number of labeled copies of $C_{\pi,4}$ in H_n is at most $p^{|E(C_{\pi,4})|} n^{|V(C_{\pi,4})|} + o(n^{|V(C_{\pi,4})|})$.*
- $\texttt{Cycle}_{4\ell}[\pi]$: *the number of labeled copies of $C_{\pi,4\ell}$ in H_n is at most $p^{|E(C_{\pi,4\ell})|} n^{|V(C_{\pi,4\ell})|} + o(n^{|V(C_{\pi,4\ell})|})$.*

The remainder of this paper is organized as follows. Section 2 contains the definitions of eigenvalues we will require from [4]. Section 3 contains definitions about linear maps and also a statement of the main technical contribution of this note. Section 4 contains the algebraic properties required for the proof of Theorem 2. Section 5 contains a crucial lemma from [4] that relates cycle counts to the trace of higher order matrices, and finally Section 6 contains the proof of Theorem 2.

2. Hypergraph eigenvalues

In this section, we give the definitions of the first and second largest eigenvalues of a hypergraph. These definitions are identical to those given in [4].

Definition 4 (*Friedman and Wigderson [2,3]*)**.** Let H be a k-uniform hypergraph with loops. The *adjacency map of H* is the symmetric k-linear map $\tau_H : W^k \to \mathbb{R}$ defined as follows, where W is the vector space over $\mathbb{R}$ of dimension $|V(H)|$. First, for all $v_1, \ldots, v_k \in V(H)$, let

$$\tau_H(e_{v_1}, \ldots, e_{v_k}) = \begin{cases} 1 & \{v_1, \ldots, v_k\} \in E(H), \\ 0 & \text{otherwise}, \end{cases}$$

where e_v denotes the indicator vector of the vertex v, that is the vector which has a one in coordinate v and zero in all other coordinates. We have defined the value of τ_H when the inputs are standard basis vectors of W. Extend τ_H to all the domain linearly.

Definition 5. Let W be a finite dimensional vector space over $\mathbb{R}$, let $\sigma : W^k \to \mathbb{R}$ be any k-linear function, and let $\vec{\pi}$ be a proper ordered partition of k, so $\vec{\pi} = (k_1, \ldots, k_t)$ for some integers $k_1, \ldots, k_t$ with $t \geq 2$. Now define a t-linear function $\sigma_{\vec{\pi}} : W^{\otimes k_1} \times \cdots \times W^{\otimes k_t} \to \mathbb{R}$ by first defining $\sigma_{\vec{\pi}}$ when the inputs are basis vectors of $W^{\otimes k_i}$ and then extending linearly. For each i, $B_i = \{b_{i,1} \otimes \cdots \otimes b_{i,k_i} : b_{i,j}$ is a standard basis vector of $W\}$ is a basis of $W^{\otimes k_i}$, so for each i, pick $b_{i,1} \otimes \cdots \otimes b_{i,k_i} \in B_i$ and define

$$\sigma_{\vec{\pi}}\left(b_{1,1} \otimes \cdots \otimes b_{1,k_1}, \ldots, b_{t,1} \otimes \cdots \otimes b_{t,k_t}\right) = \sigma(b_{1,1}, \ldots, b_{1,k_1}, \ldots, b_{t,1}, \ldots, b_{t,k_t}).$$

Now extend $\sigma_{\vec{\pi}}$ linearly to all of the domain. $\sigma_{\vec{\pi}}$ will be t-linear since σ is k-linear.