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Vertex-coloring 3-edge-weighting of some graphs

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ABSTRACT

Let G be a non-trivial graph and $k \in \mathbb{Z}^+$. A vertex-coloring k-edge-weighting is an assignment $f : E(G) \to \{1, \ldots, k\}$ such that the induced labeling $f : V(G) \to \mathbb{Z}^+$, where $f(v) = \sum_{e \in E(v)} f(e)$ is a proper vertex coloring of G. It is proved in this paper that *every 4-edge-connected graph with chromatic number at most 4 admits a vertex-coloring 3-edge-weighting.*

1. Introduction

For technical reasons, all graphs considered in this paper are connected multigraphs with parallel edges but no loop. A graph with two vertices and m parallel edges is denoted by mK_2.

Let G be a graph. A vertex-coloring k-edge-weighting of G is an assignment $f : E(G) \to \{1, \ldots, k\}$ such that the induced labeling $f : V(G) \to \mathbb{Z}^+$, where $f(v) = \sum_{e \in E(v)} f(e)$, is a proper vertex coloring of G (see [1,2,3,6,11], or a comprehensive survey paper [9]).

In [6], Karoński, Luczak and Thomason conjectured *(the 1-2-3-conjecture) that every graph other than mK_2 admits a vertex coloring 3-edge-weighting*. It is proved in [5] that every graph other than mK_2 admits a vertex-coloring 5-edge-weighting. It also proved in [6] that every 3-colorable graph other than mK_2 admits a vertex-coloring 3-edge-weighting; and in [7] that every 4-colorable graph other than mK_2 admits a vertex-coloring 4-edge-weighting. In this paper, we extend some of these results by verifying the 1-2-3-conjecture for some graphs G with $\chi(G) \le 4$.

Theorem 1.1. *Every 4-edge-connected 4-colorable multigraph G admits a vertex-coloring 3-edge-weighting.*

1.1. Notation and terminology

We follow [4] and [12] for terms and notation.

A circuit is a connected 2-regular graph.

Let H_1 and H_2 be two subgraphs of a graph G. The symmetric difference of H_1 and H_2, denoted by $H_1 \triangle H_2$, is the subgraph of G induced by the set of edges $[E(H_1) \cup E(H_2)] \setminus [E(H_1) \cap E(H_2)]$.

Let G be a graph. The set of odd vertices of G is denoted by $O(G)$. Let U be a subset of $V(G)$ with even order. A spanning subgraph Q is called T-join of G (with respected to U) if $O(Q) = U$.

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2. Proof of the main theorem

2.1. Sketch of an outline of the proof

Let $\beta : V(G) \to Z_4$ be a 4-coloring of G. We are to find a vertex-coloring 3-edge-weighting f such that $f(v) \equiv \beta(v) \pmod 4$ for every vertex v.

A necessary condition of β is $\sum_{v\in V(G)}\beta(v) \equiv 0 \pmod 2$ since

$$\sum_{v\in V(G)} f(v) \equiv 2\sum_{e\in E(G)} f(e) \equiv 0 \pmod 2.$$

Let

$$W_\mu = \{v \in V(G) : \ \beta(v) - d_G(v) \equiv \mu \pmod 4\}.$$

The first step of the proof is to find a T-join Q with $O(Q) = W_1 \cup W_3$. Define $g : E(Q) \to \{2\}$, and $\beta' : V(G) \to Z_4$ such that

$$\beta'(x) \equiv \left\{\begin{matrix} \beta(x) - 2 & \text{if } x \in W_1 \cup W_3 \\ \beta(x) & \text{otherwise} \end{matrix}\right\} \pmod 4.$$

It will be proved that, in the subgraph $G - E(Q)$, $d_{G-E(Q)}(v) \equiv \beta'(v)$ or $\beta'(v) + 2 \pmod 4$ for every $v \in V(G)$.

In the second step, Lemma 2.2 is applied to find another edge-weight $f_0 : E(G) - E(Q) \to \{1, 3\}$ such that, for every $v \in V(G)$,

$$\beta'(v) \equiv \sum_{e\in E(v)-E(Q)} f_0(e) \pmod 4.$$

Thus, the combination of g and f_0 yields a vertex-coloring 3-edge-weighting of G.

By Tutte and Nash-Williams Theorem [8,10], a 4-edge-connected graph contains a pair of edge-disjoint spanning trees T_1, T_2. The subset Q is to be found in $G - T_2$, and the weight f_0 is assigned in $E(G) - E(Q)$. We notice that a straightforward application of Tutte–Nash-Williams Theorem is not sufficient due to a parity requirement for $|Q|$. Thus, Tutte–Nash-Williams Theorem is extended in Lemma 2.1 in order to meet the requirements of Lemma 2.2 in the second step of the proof.

2.2. Lemmas

Lemma 2.1. *If G is a 4-edge-connected non-bipartite graph, then $E(G)$ has a partition $\{T_1, T_2, F\}$ such that each T_i is a spanning tree and $T_1 + F$ contains an odd-circuit.*

Lemma 2.2. *Let H be a graph and let $\beta_H : V(H) \to Z_4$ be a mapping. Assume that*

(i) *H is connected;*
(ii) $\beta_H(v) \equiv d_H(v) \pmod 2$ *for each vertex $v \in V(H)$;*
(iii) $\sum_{v\in V(H)}\beta_H(v) \equiv 2|E(H)| \pmod 4$.

Then there exists a mapping $f_H : E(H) \to \{1, 3\}$ such that for each vertex $x \in V(H)$,

$$f_H(x) = \sum_{e\in E(x)} f_H(e) \equiv \beta_H(x) \pmod 4. \tag{1}$$

See Section 3 for proofs of both lemmas.

2.3. Proof of Theorem 1.1

We pay only attention to graphs with chromatic number $\chi = 4$ since it was proved in [6] that every multigraph G with $\chi(G) \le 3$ admits a vertex-coloring 3-edge-weighting.

I. Since $\chi(G) = 4$, there exists a vertex partition $\{V_0, V_1, V_2, V_3\}$ of $V(G)$ such that each V_i is an independent set, $i = 0, 1, 2$ and 3. Renaming them if necessary, we can assume that $|V_1| + |V_3|$ is even. Define $\beta : V(G) \to \{0, 1, 2, 3\}$ such that $\beta(v) = i$ if $v \in V_i$. Then

$$\sum_{x\in V(G)} \beta(x) = |V_1| + 2|V_2| + 3|V_3| \equiv |V_1| + |V_3| \equiv 0 \pmod 2. \tag{2}$$

Our goal is to find an edge-weighting $f : E(G) \to \{1, 2, 3\}$ such that

$$\sum_{e\in E(x)} f(e) \equiv \beta(x) \pmod 4 \tag{3}$$

for every vertex x.