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Total monochromatic connection of graphs☆

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ABSTRACT

A graph is said to be *total-colored* if all the edges and the vertices of the graph are colored. A path in a total-colored graph is a *total monochromatic path* if all the edges and internal vertices on the path have the same color. A total-coloring of a graph is a *total monochromatically-connecting coloring* (*TMC-coloring*, for short) if any two vertices of the graph are connected by a total monochromatic path of the graph. For a connected graph G, the *total monochromatic connection number*, denoted by $\text{tmc}(G)$, is defined as the maximum number of colors used in a TMC-coloring of G. These concepts are inspired by the concepts of monochromatic connection number $\text{mc}(G)$, monochromatic vertex connection number $\text{mvc}(G)$ and total rainbow connection number $\text{trc}(G)$ of a connected graph G. Let $l(T)$ denote the number of leaves of a tree T, and let $l(G) = \max\{l(T)|\ T \text{ is a spanning tree of } G\}$ for a connected graph G. In this paper, we show that there are many graphs G such that $\text{tmc}(G) = m - n + 2 + l(G)$, and moreover, we prove that for almost all graphs G, $\text{tmc}(G) = m - n + 2 + l(G)$ holds. Furthermore, we compare $\text{tmc}(G)$ with $\text{mvc}(G)$ and $\text{mc}(G)$, respectively, and obtain that there exist graphs G such that $\text{tmc}(G)$ is not less than $\text{mvc}(G)$ and vice versa, and that $\text{tmc}(G) = \text{mc}(G) + l(G)$ holds for almost all graphs. Finally, we prove that $\text{tmc}(G) \leq \text{mc}(G) + \text{mvc}(G)$, and the equality holds if and only if G is a complete graph.

1. Introduction

In this paper, all graphs are simple, finite and undirected. We refer to the book [3] for undefined notation and terminology in graph theory. Throughout this paper, let n and m denote the order (number of vertices) and size (number of edges) of a graph, respectively. Moreover, a vertex of a connected graph is called a *leaf* if its degree is one; otherwise, it is called an *internal vertex*. Let $l(T)$ and $q(T)$ denote the number of leaves and the number of internal vertices of a tree T, respectively, and let $l(G) = \max\{l(T)|\ T \text{ is a spanning tree of } G\}$ and $q(G) = \min\{q(T)|\ T \text{is a spanning tree of } G\}$ for a connected graph G. Note that the sum of $l(G)$ and $q(G)$ is n for any connected graph G of order n. A path in an edge-colored graph is a *monochromatic path* if all the edges on the path have the same color. An edge-coloring of a connected graph is a *monochromatically-connecting coloring* (*MC-coloring*, for short) if any two vertices of the graph are connected by a monochromatic path of the graph. For a connected graph G, the *monochromatic connection number* of G, denoted by $\text{mc}(G)$, is defined as the maximum number of colors used in an MC-coloring of G. An *extremal MC-coloring* is an MC-coloring that uses $\text{mc}(G)$ colors. Note that $\text{mc}(G) = m$ if and only if G is a complete graph. The concept of $\text{mc}(G)$ was first introduced by Caro and Yuster [6] and has been well-studied recently. We refer the reader to [4,8] for more details.

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As a natural counterpart of the concept of monochromatic connection, Cai et al. [5] introduced the concept of monochromatic vertex connection. A path in a vertex-colored graph is a *vertex-monochromatic path* if its internal vertices have the same color. A vertex-coloring of a graph is a *monochromatically-vertex-connecting coloring* (*MVC-coloring*, for short) if any two vertices of the graph are connected by a vertex-monochromatic path of the graph. For a connected graph G, the *monochromatic vertex connection number*, denoted by $\mathrm{mvc}(G)$, is defined as the maximum number of colors used in an MVC-coloring of G. An *extremal MVC-coloring* is an MVC-coloring that uses $\mathrm{mvc}(G)$ colors. Note that $\mathrm{mvc}(G) = n$ if and only if $diam(G) \leq 2$.

Actually, the concepts of monochromatic connection number $\mathrm{mc}(G)$ and monochromatic vertex connection number $\mathrm{mvc}(G)$ are natural opposite concepts of rainbow connection number $\mathrm{rc}(G)$ and rainbow vertex connection number $\mathrm{rvc}(G)$. For details about them we refer the readers to the book [10] and the survey paper [9]. The concept of total rainbow connection number $\mathrm{trc}(G)$ in [12] was motivated by the rainbow connection number $\mathrm{rc}(G)$ and rainbow vertex connection number $\mathrm{rvc}(G)$. Naturally, here we introduce the concept of total monochromatic connection of graphs. A graph is said to be *total-colored* if all the edges and the vertices of the graph are colored. A path in a total-colored graph is a *total monochromatic path* if all the edges and internal vertices on the path have the same color. A total-coloring of a graph is a *total monochromatically-connecting coloring* (*TMC-coloring*, for short) if any two vertices of the graph are connected by a total monochromatic path of the graph. For a connected graph G, the *total monochromatic connection number*, denoted by $\mathrm{tmc}(G)$, is defined as the maximum number of colors used in a TMC-coloring of G. An *extremal TMC-coloring* is a TMC-coloring that uses $\mathrm{tmc}(G)$ colors. It is easy to check that $\mathrm{tmc}(G) = m + n$ if and only if G is a complete graph.

The rest of this paper is organized as follows: In Section 2, we prove that $\mathrm{tmc}(G) \geq m - n + 2 + l(G)$ for any connected graph and determine the value of $\mathrm{tmc}(G)$ for some special graphs. In Section 3, we prove that there are many graphs with $\mathrm{tmc}(G) = m - n + 2 + l(G)$ which are restricted by other graph parameters such as the maximum degree, the diameter and so on. Moreover, we show that for almost all graphs G, $\mathrm{tmc}(G) = m - n + 2 + l(G)$ holds. In Section 4, we compare $\mathrm{tmc}(G)$ with $\mathrm{mvc}(G)$ and $\mathrm{mc}(G)$, respectively, and obtain that there exist graphs G such that $\mathrm{tmc}(G)$ is not less than $\mathrm{mvc}(G)$ and vice versa, and that $\mathrm{tmc}(G) = \mathrm{mc}(G) + l(G)$ for almost all graphs. We also prove that $\mathrm{tmc}(G) \leq \mathrm{mc}(G) + \mathrm{mvc}(G)$, and the equality holds if and only if G is a complete graph.

2. Preliminary results

In this section, we show that $\mathrm{tmc}(G) \geq m-n+2+l(G)$ and present some preliminary results on the total monochromatic connection number. Moreover, we determine the value of $\mathrm{tmc}(G)$ when G is a tree, a wheel, and a complete multipartite graph. It is easy to see the following fact.

Proposition 1. *If G is a connected graph and H is a connected spanning subgraph of G, then* $\mathrm{tmc}(G) \geq e(G) - e(H) + \mathrm{tmc}(H)$.

Since for any two vertices of a tree, there exists only one path connecting them, we have the following result.

Proposition 2. *If T is a tree, then* $\mathrm{tmc}(T) = l(T) + 1$.

The consequence below is immediate from Propositions 1 and 2.

Theorem 1. *For a connected graph G,* $\mathrm{tmc}(G) \geq m - n + 2 + l(G)$.

Next we give an important and useful property of an extremal TMC-coloring.

Fact 1. *Let G be a connected graph and f be an extremal TMC-coloring of G that uses a given color c. Then the subgraph H formed by the edges and vertices colored c is a tree whose each internal vertex is colored c.*

Proof. We first claim that H is connected. Otherwise, we will give a fresh color to all the edges and vertices colored c in some component of H while still maintaining a TMC-coloring of G, contradicting the assumption on f. Before proving that H is acyclic, we show that the color of each internal vertex of H is c. Let $u_1, \ldots, u_t$ be the internal vertices of H such that each of them is not colored c. We obtain the subgraph H_0 of H by deleting the vertices $\{u_1, \ldots, u_t\}$. If H_0 is connected, it is possible to choose an edge incident with u_1 in H and assign it with a fresh color while still maintaining a TMC-coloring of G, a contradiction. If not, we can give a fresh color to all the edges and vertices colored c in some component of H_0 while still maintaining a TMC-coloring of G, a contradiction. Now we prove that H does not contain any cycle. Suppose that H has a cycle, say C. Then a fresh color can be assigned to any edge of the cycle C while still maintaining a TMC-coloring of G, which contradicts the assumption on f.

Thus, H is a tree whose each internal vertex is colored c. □

Let G be a connected graph and f be an extremal TMC-coloring of G that uses a given color c. Now we define the *color tree* as the tree formed by the edges and vertices colored c, denoted by T_c. If T_c has at least two edges, the color c is called *nontrivial*. Otherwise, c is *trivial*. We call an extremal TMC-coloring *simple* if for any two nontrivial colors c and d, the corresponding trees T_c and T_d intersect in at most one vertex. If f is simple, then the leaves of T_c must have distinct colors different from color c. Otherwise, we can give a fresh color to such a leaf while still maintaining a TMC-coloring. Moreover, a nontrivial