Discrete Mathematics 340 (2017) 209–222

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Tolerance intersection graphs of degree bounded subtrees of a tree with constant tolerance 2

Elad Cohen [a,*], Martin Charles Golumbic [b,a], Marina Lipshteyn [b], Michal Stern [c,b]

[a] Department of Computer Science, University of Haifa, Mount Carmel, 31905, Israel
[b] Caesarea Rothschild Institute, University of Haifa, Mount Carmel, 31905, Israel
[c] The Academic College of Tel-Aviv - Yaffo, Rabeno Yeruham 2, 68182, Tel-Aviv, Israel

ARTICLE INFO

Article history:
Received 29 May 2011
Received in revised form 12 August 2016
Accepted 14 August 2016

Keywords:
Chordal graph
Weakly chordal graph
Intersection graph of subtrees of a tree
Complete bipartite graph

ABSTRACT

An (h, s, t)-representation of a graph G consists of a collection of subtrees $\{S_v : v \in V(G)\}$ of a tree T, such that (i) the maximum degree of T is at most h, (ii) every subtree has maximum degree at most s, and (iii) there is an edge between two vertices in the graph if and only if the corresponding subtrees in T have at least t vertices in common. Jamison and Mulder denote the family of graphs that admit such a representation as $[h, s, t]$.

Our main theorem shows that the class of weakly chordal graphs is incomparable with the class $[h, s, t]$. We introduce new characterizations of the graph $K_{2,n}$ in terms of the families $[h, s, 2]$ and $[h, s, 3]$. We then present our second main result characterizing the graphs in $[4, 3, 2]$ as being the graphs in $[4, 4, 2]$ avoiding a particular family of substructures, and we give a recognition algorithm for the family $[4, 3, 2]$.

1. Introduction and motivation

A graph is *chordal* if it contains no chordless cycle of length greater than 3. The class of chordal graphs is widely investigated and plays a fundamental role in graph theory. One of the reasons is that chordal graphs can be modeled as intersection graphs of subtrees of a tree (c.f. [2,5] and [18]). One can construct in linear time such an intersection representation, called a "clique tree", such that each node of the host tree corresponds to a maximal clique in the chordal graph.

In many real world applications, the intersection representation of a graph is more important than the graph itself. In [15] and [16], the intersection representation of a graph on a tree is defined as follows. An (h, s, t)-representation $\langle \mathcal{S}, T \rangle$ of G consists of a collection of subtrees $\mathcal{S} = \{S_v : v \in V(G)\}$ of a tree T, such that (i) the maximum degree of T is at most h, (ii) every subtree has maximum degree at most s, and (iii) there is an edge between two vertices in G if and only if the corresponding subtrees in T have at least t vertices in common. The class of graphs that have an (h, s, t)-representation is denoted by $[h, s, t]$. Throughout the paper we will use the following definition.

Definition 1.1. We say that G is *sharply contained* in $[h, s, t]$ if $G \in [h, s, t]$, $G \notin [h', \infty, t]$ and $G \notin [\infty, s', t]$, where $h' < h$ and $s' < s$, respectively.

Thus, the family of chordal graphs is the same as $[\infty, \infty, 1]$. This result was strengthened in [17] and [16], respectively, showing that the family of chordal graphs is also equivalent to $[3, 3, 1]$ and $[3, 3, 2]$. The notation ∞ here means that no

* Corresponding author. Fax: +972 4 8288181.
E-mail addresses: eladdc@gmail.com (E. Cohen), golumbic@cs.haifa.ac.il (M.C. Golumbic), lipshteynmarina@gmail.com (M. Lipshteyn), stern@mta.ac.il (M. Stern).

http://dx.doi.org/10.1016/j.disc.2016.08.015

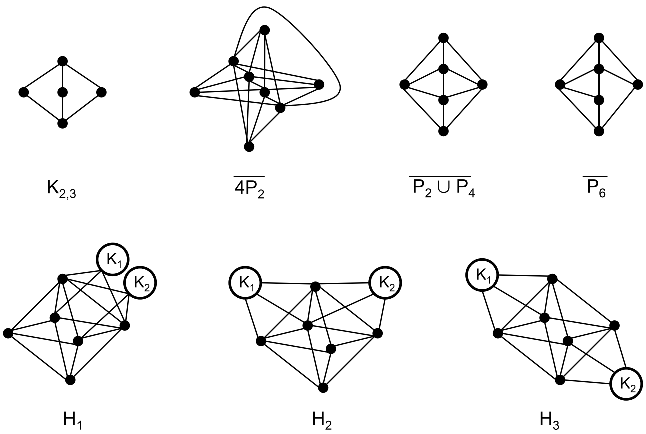

Fig. 1. The forbidden subgraphs from Theorem 1.2.

restriction is imposed. The family of interval graphs, by definition, is the family [2, 2, 1] and is equivalent to [2, 2, 2]. There are other papers that study $[h, s, t]$, for specific values of h, s and t, although without using this notion. For example, the family of edge intersection graphs of paths in a tree [7] (EPT graphs) is the class $[\infty, 2, 2]$, and the family of vertex intersection graphs of paths in a tree [6,13] (VPT graphs or path graphs) is $[\infty, 2, 1]$.

A graph is *weakly chordal* if neither the graph nor its complement contains a chordless cycle of length greater than 4. The class of weakly chordal graphs is also well studied and has a number of known applications [1,4,12,14]. It was shown in [10] that [4, 2, 2] is equivalent to the intersection of the family of weakly chordal graphs and $[\infty, 2, 2]$, and in [8,12] the following theorem characterizing the graphs in [4, 4, 2] was given.

Theorem 1.2 (*[8,12]*). *A graph G is a weakly chordal ($K_{2,3}$, $\overline{4P_2}$, $\overline{P_2 \cup P_4}$, $\overline{P_6}$, H_1, H_2, H_3)-free graph (see Fig. 1) if and only if the graph G has a* (4, 4, 2)*-representation.*

Our main motivation in this paper has been to determine if there is an $[h, s, t]$ class of graphs that corresponds to the class of weakly chordal graphs. From our results on the complete bipartite graphs, which are a family of weakly chordal graphs, we are able to confirm that weakly chordal graphs cannot be characterized within the $[h, s, t]$ framework. Our second important goal has been to characterize the class [4, 3, 2] which is the last open case for the families $[h, s, t]$ with $4 \geq h \geq s$ and $t \geq 2$.

In Section 2, we investigate the complete bipartite graph $K_{2,n}$, where one part of the bipartition has two vertices and the other part has n vertices. In [15] and [16], a function $f(h, s, t)$ is given, such that for any $n > f(h, s, t)$, the graph $K_{2,n}$ has no (h, s, t)-representation. We strengthen their results and prove new theorems characterizing n such that $K_{2,n}$ has an $(h, s, 2)$-representation and those such that $K_{2,n}$ has an $(h, s, 3)$-representation.

In Section 3, we characterize the family [4, 3, 2]. In particular, we introduce a new family of graphs called "4-flowers", we prove that a 4-flower graph is sharply contained in [4, 4, 2] and that [4, 3, 2] is the family of 4-flower-free graphs belonging to [4, 4, 2]. We also provide a polynomial-time recognition algorithm for the class [4, 3, 2] in Section 5. The recognition algorithm is used to prove the characterization theorem. In Section 4, we show a hierarchy of families of graphs between chordal and weakly chordal within the $[h, s, t]$ framework. The hierarchy also includes all cases of $[\infty, s, 2]$.

2. The (h, s, t)-representations of $K_{2,n}$

Jamison and Mulder (c.f. [15] and [16]) investigated the intersection graph of subtrees of a tree by studying the representations of the complete bipartite graph $K_{2,n}$. They found an upper bound for the size of n as a function of s and t, but they observed that this bound is far from being optimal.

Let $R(s, t)$ denote the complete balanced rooted tree whose root has s children, internal nodes have $s - 1$ children and all leaves are at distance $t - 1$ from the root. Let $\gamma(s, t)$ be the number of subtrees of $R(s, t)$ which have exactly t nodes and which contain the root. Jamison and Mulder prove in [16] the following:

Theorem 2.1 (*[16]*). *If h, s and t are positive integers with $h \geq s$, and n is a positive integer with $n > \gamma(s, t)(t + 1)$, then $K_{2,n} \notin [h, s, t]$.*

In this section, we will improve the bound of Jamison and Mulder for $t = 2$ and $t = 3$. The case of $t = 2$ will be used in Section 4 for particular separation examples in the hierarchy shown in Fig. 5. The case of $t = 3$ continues the search of Jamison and Mulder for (h, s, t)-representations for the graph $K_{2,n}$.

We introduce the following notation for an (h, s, t)-representation of $K_{2,n}$. Let $\{a, b\}$ be the vertices of the 2-side of $K_{2,n}$, and let A and B be the subtrees of the host tree T that correspond to a and b. Similarly, let $\{s_i\}$ be the vertices of the n-side of $K_{2,n}$, and $\mathcal{S}' = \{S_i\}$ the subtrees on the host tree T that correspond to $\{s_i\}$ respectively, for $1 \leq i \leq n$.