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# The complexity of signed graph and edge-coloured graph homomorphisms

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ABSTRACT

We study homomorphism problems of signed graphs from a computational point of view. A signed graph $(G, \Sigma)$ is a graph $G$ where each edge is given a sign, positive or negative; $\Sigma \subseteq E(G)$ denotes the set of negative edges. Thus, $(G, \Sigma)$ is a 2-edge-coloured graph with the property that the edge-colours, $\{+, -\}$, form a group under multiplication. Central to the study of signed graphs is the operation of switching at a vertex, that results in changing the sign of each incident edge. We study two types of homomorphisms of a signed graph $(G, \Sigma)$ to a signed graph $(H, \Pi)$: ec-homomorphisms and s-homomorphisms. Each is a standard graph homomorphism of $G$ to $H$ with some additional constraint. In the former, edge-signs are preserved. In the latter, edge-signs are preserved after the switching operation has been applied to a subset of vertices of $G$.

We prove a dichotomy theorem for s-homomorphism problems for a large class of (fixed) target signed graphs $(H, \Pi)$. Specifically, as long as $(H, \Pi)$ does not contain a negative (respectively a positive) loop, the problem is polynomial-time solvable if the core of $(H, \Pi)$ has at most two edges, and is NP-complete otherwise. (Note that this covers all simple signed graphs.) The same dichotomy holds if $(H, \Pi)$ has no negative digons, and we conjecture that it holds always. In our proofs, we reduce s-homomorphism problems to certain ec-homomorphism problems, for which we are able to show a dichotomy. In contrast, we prove that a dichotomy theorem for ec-homomorphism problems (even when restricted to bipartite target signed graphs) would settle the dichotomy conjecture of Feder and Vardi.



## 1. Introduction and terminology

Graph homomorphisms and their variants play a fundamental role in the study of computational complexity. For example, the celebrated CSP Dichotomy Conjecture of Feder and Vardi [8], a major open problem in the area, can be reformulated in terms of digraph homomorphisms or graph retractions (to fixed targets). As a special case, the dichotomy theorem of Hell and Nešetřil [14] shows that there are no NP-intermediate graph homomorphism problems. In this paper, we study homomorphisms of signed graphs from an algorithmic point of view. We study two natural types of homomorphism problems on signed graphs, one with switching and one without. For the former, we prove a dichotomy theorem for a large

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class of signed graphs, while for the latter we prove that a dichotomy would answer the CSP Dichotomy Conjecture in the positive.

We begin by defining signed graphs and the two types of homomorphisms. We remark that we have adopted the language of signed graphs for this paper, but readers familiar with edge-coloured graphs will recognize that our work may be equivalently formulated in terms of edge-coloured graphs. For example, the edge-coloured viewpoint is used in [25].

### 1.1. Signed graphs

A *signed graph* is a graph $G$ together with a signing function $\sigma : E(G) \to \{+, -\}$. By setting $\Sigma = \sigma^{-1}(-)$, we use the notation $(G, \Sigma)$ to denote this signed graph. The set $\Sigma$ of negative edges is referred to as the *signature* of $(G, \Sigma)$. In all our diagrams, these edges are drawn in red with dashed lines. The other edges, which are positive, are drawn in blue with solid lines. Signed graphs were introduced by Harary in [12], and studied in depth by Zaslavsky (see for example [26–30]). The notion of distinguishing a set $\Sigma$ of edges can also be found in the work of König [18].

Signed graphs are different from 2-edge-coloured graphs with arbitrary colours, due to the fact that $\{+, -\}$ forms a group with respect to the product of signs. The most crucial of these differences comes from the following definition of the *sign* of a cycle, or more generally, a closed walk. A cycle or closed walk of $(G, \Sigma)$ is said to be *negative* if the product of the signs of all the edges (considering multiplicities if an edge is traversed more than once) is the negative sign, and *positive* otherwise. A (signed) subgraph of $(G, \Sigma)$ is called *balanced* if it contains no negative cycle (equivalently no negative closed walk, noting that each negative closed walk must contain a negative cycle). This notion of balance was introduced by Harary [12] and a similar idea appears in the work of König [18]. A cycle of length 2 is a *digon*. In our work, we do not consider multiple edges of the same sign, and thus we only consider negative digons.

The second notion of importance for signed graphs is the operation of *switching*, introduced by Zaslavsky [27]. To *switch* at a vertex $v$ means to multiply the signs of all edges incident to $v$ by $-$, that is, to switch the sign of each of these edges. (In the case of a loop at $v$, its sign is multiplied twice and hence it is invariant under switching.) Given signatures $\Sigma$ and $\Sigma'$ on a graph $G$, the signature $\Sigma'$ is said to be *switching equivalent* to $\Sigma$, denoted $\Sigma \equiv \Sigma'$, if it can be obtained from $\Sigma$ by a sequence of switchings. Equivalently, $\Sigma \equiv \Sigma'$ if their symmetric difference is an edge-cut of $G$.

Zaslavsky proved that two signatures $\Sigma$ and $\Sigma'$ of a graph $G$ are switching equivalent if and only if they induce the same cycle signs [27]. Inherent in the proof is an algorithm to test whether $\Sigma$ and $\Sigma'$ are switching equivalent. For completeness, in Section 2.1, we offer an alternative certifying algorithm obtained by generalizing a method from [12].

Each of the two types of homomorphisms studied in this paper capture, in particular, the concept of proper vertex-colouring of signed graphs introduced by Zaslavsky [28] (as mappings to certain families of signed graphs).

### 1.2. ec-homomorphisms

Recall that given two graphs $G$ and $H$, a *homomorphism* $\varphi$ of $G$ to $H$ is a mapping of the vertices $\varphi : V(G) \to V(H)$ such that if two vertices $x$ and $y$ are adjacent in $G$, then their images are adjacent in $H$. We write $G \to H$ to denote the existence of a homomorphism or $\varphi : G \to H$ when we wish to explicitly name the mapping. One natural extension of this idea to signed graphs is to additionally require that homomorphisms preserve the sign of edges.

**Definition 1.1.** Let $(G, \Sigma)$ and $(H, \Pi)$ be two signed graphs. An *ec-homomorphism* of $(G, \Sigma)$ to $(H, \Pi)$ is a (graph) homomorphism $\varphi : G \to H$ such that for each edge $e$ between two vertices $x$ and $y$ in $(G, \Sigma)$, there is an edge between $\varphi(x)$ and $\varphi(y)$ in $(H, \Pi)$ having the same sign as $e$.

When there exists such an ec-homomorphism, we write $(G, \Sigma) \xrightarrow{ec} (H, \Pi)$ or $\varphi : (G, \Sigma) \xrightarrow{ec} (H, \Pi)$ when we wish to explicitly name the mapping.

An ec-homomorphism $r : (G, \Sigma) \xrightarrow{ec} (H, \Pi)$ is an *ec-retraction* if $(H, \Pi)$ is a subgraph of $(G, \Sigma)$ and $r$ is the identity on $(H, \Pi)$. A signed graph $(H, \Pi)$ is an *ec-core* if for each ec-homomorphism $\varphi : (H, \Pi) \xrightarrow{ec} (H, \Pi)$, the mapping $\varphi$ is an ec-automorphism. Every signed graph $(H, \Pi)$ admits an ec-retraction to a subgraph $(H', \Pi')$ that is an ec-core. In fact, $(H', \Pi')$ is unique up to ec-isomorphism and we call it *the ec-core* of $(H, \Pi)$ [15].

The complexity of determining the existence of homomorphisms has received much attention in the literature. For classical undirected graphs, the complexity (for fixed targets) is completely determined by the dichotomy theorem of Hell and Nešetřil. Let $H$ be a fixed graph. We define the decision problem Hom($H$), also known as $H$-Colouring.

Hom($H$)
Instance: A graph $G$.
Question: Does $G \to H$?

**Theorem 1.2** (*Hell and Nešetřil [14]*)**.** *If a graph $H$ is bipartite or contains a loop, then* Hom($H$) *is polynomial-time solvable; otherwise, it is NP-complete.*