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Spanning trails with variations of Chvátal–Erdős conditions

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ABSTRACT

Let $\alpha(G)$, $\alpha'(G)$, $\kappa(G)$ and $\kappa'(G)$ denote the independence number, the matching number, connectivity and edge connectivity of a graph G, respectively. We determine the finite graph families $\mathcal{F}_1$ and $\mathcal{F}_2$ such that each of the following holds.

(i) If a connected graph G satisfies $\kappa'(G) \geq \alpha(G) - 1$, then G has a spanning closed trail if and only if G is not contractible to a member of $\mathcal{F}_1$.

(ii) If $\kappa'(G) \geq \max\{2, \alpha(G) - 3\}$, then G has a spanning trail. This result is best possible.

(iii) If a connected graph G satisfies $\kappa'(G) \geq 3$ and $\alpha'(G) \leq 7$, then G has a spanning closed trail if and only if G is not contractible to a member of $\mathcal{F}_2$.

1. Introduction

In this paper, graphs considered are finite and loopless. We follow [5] for undefined terms and notation. Let $\boldsymbol{N_G(u)}$ be the set of vertices adjacent to u in G, and $\boldsymbol{D_i(G) = \{v \in V(G) : d(v) = i\}}$. As in [5], for a graph G, let $\boldsymbol{\alpha(G)}$, $\boldsymbol{\alpha'(G)}$, $\boldsymbol{\kappa(G)}$, $\boldsymbol{\kappa'(G)}$ denote independence number, matching number, connectivity and edge connectivity of G, respectively. An edge cut E of a graph G is **essential** if $G - E$ contains two nontrivial components. We use $\boldsymbol{O(G)}$ to denote the set of all odd degree vertices of G. A cycle on n vertices is often called an ***n*-cycle**. For $A \subseteq V(G) \cup E(G)$, $\boldsymbol{G[A]}$ is the subgraph of G induced by A, and $\boldsymbol{G - A}$ is the subgraph of G obtained by deleting the elements in A. Let H be a graph. We say G is ***H*-free** if G does not contain H as a subgraph.

As in [5], a graph G is **eulerian** if G is a closed trail. Equivalently, G is eulerian if G is connected with $O(G) = \emptyset$. A graph is **supereulerian** if it has a spanning closed trail. Boesch et al. [3] first posed the problem of characterizing supereulerian graphs. Pulleyblank [19] proved that determining if a 3-edge-connected planar graph is supereulerian is NP-complete. Catlin [8] gave a survey on supereulerian graphs, which was supplemented and updated in [14,18].

Motivated by a well-known result of Chvátal and Erdős [15] that every graph G with $\kappa(G) \geq \alpha(G)$ is Hamiltonian, there have been researches on conditions analogous to this Chvátal–Erdős Theorem to assure the existence of spanning trials in a graph utilizing relationship among independence number, matching number and edge-connectivity. See [1,16,17] and [21], among others. Let $\boldsymbol{P(10)}$ denote the Petersen graph and let $\boldsymbol{K_{2,3}(1,2,2)}$, $\boldsymbol{S_{1,2}}$, $\boldsymbol{K'_{2,3}}$ be the graphs depicted in Fig. 1. Let $\boldsymbol{P^n}$ be a path of order n. Define

$$\mathcal{F}_1 = \{K_2, P^3, P^4, K_{2,3}, K_{2,3}(1,2,2), S_{1,2}, P(10)\} \quad \text{and} \quad \mathcal{F}_2 = \{P(10), P(14)\}.$$

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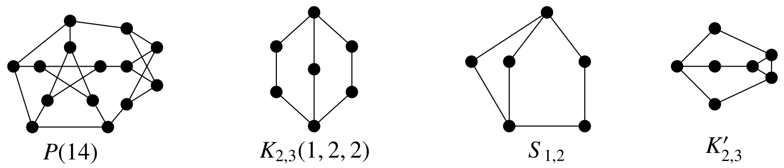

Fig. 1. $P(14)$ and some graphs in $\mathcal{F}_1$.

Theorem 1.1 (*Han et al., Theorem 3 of [16]*). *Let G be a simple graph with $\kappa(G) \geq 2$. If $\kappa(G) \geq \alpha(G) - 1$, then exactly one of the following holds.*

(i) *G is supereulerian.*

(ii) *$G \in \{P(10), K_{2,3}, K_{2,3}(1,2,2), S_{1,2}, K'_{2,3}\}$.*

(iii) *G is a 2-connected graph obtained from $K_{2,3}$ (resp. $S_{1,2}$) by replacing a vertex whose neighbors have degree three in $K_{2,3}$ (resp. $S_{1,2}$) with a complete graph of order at least three.*

Theorem 1.2 (*Tian and Xiong, Theorem 4 of [21]*). *If G is a 2-connected graph with $\alpha(G) \leq \kappa(G) + 3$, then G has a spanning trail.*

The supereulerian property for graphs G with $\alpha'(G) \leq 2$ and $\kappa'(G) \geq 2$ has been completely determined in [1] and [17].

The purpose of this paper is to investigate the existence of spanning trails in graphs with given relationship between independence number and edge-connectivity, or matching number with edge-connectivity. In this paper, we determine the finite graph families $\mathcal{F}_1$ and $\mathcal{F}_2$ such that each of the following holds.

Theorem 1.3. *If a graph G satisfies $\kappa'(G) \geq \alpha(G) - 1$, then G has a spanning closed trail if and only if G is not contractible to a member of $\mathcal{F}_1$.*

Theorem 1.4. *If $\kappa'(G) \geq \max\{2, \alpha(G) - 3\}$, then G has a spanning trail.*

Theorem 1.5. *If a graph G satisfies $\kappa'(G) \geq 3$ and $\alpha'(G) \leq 7$, then G has a spanning closed trail if and only if G is not contractible to a member of $\mathcal{F}_2$.*

In Section 2, we display the mechanism we will use in our arguments. In the subsequent sections, we prove the main results.

2. Preliminaries

For a subset $Y \subseteq E(G)$, the **contraction** G/Y is the graph obtained from G by identifying the two ends of each edge in Y and then by deleting the resulting loops. If H is a subgraph of G, we often use G/H for $G/E(H)$. A graph G is called **collapsible** if for any $R \subseteq V(G)$ with $|R|$ is even, G has a spanning subgraph S_R with $O(S_R) = R$. By definition, collapsible graphs are supereulerian. In [7], Catlin showed that every graph G has a unique collection of maximal collapsible subgraphs $H_1, H_2, \ldots, H_c$. The **reduction** of G, denoted by G', is the graph $G/(H_1 \cup H_2 \cup \cdots \cup H_c)$. A graph G is reduced if $G' = G$.

Theorem 2.1 (*Catlin, Theorem 2 of [7]*). *Every graph G with $\kappa'(G) \geq 4$ is collapsible.*

Theorem 2.2 (*Catlin, Theorem 3 of [7]*). *Let G be a connected graph, H be a collapsible subgraph of G and let G' be the reduction of G. Then*

(i) *G is collapsible if and only if G/H is collapsible.*

(ii) *G is supereulerian if and only if G/H is supereulerian.*

(iii) *G has a spanning trail if and only if G/H has a spanning trail.*

(iv) *Any subgraph of a reduced graph is reduced.*

Let $\boldsymbol{F(G)}$ be the minimum number of extra edges that must be added to G so that the resulting graph has two edge-disjoint spanning trees. The following results on the structures of reduced graphs will be needed.

Theorem 2.3. *Let G be a connected reduced graph. Then*

(i) *(Catlin, Theorem 7 of [6]) If $|V(G)| \geq 3$, then $F(G) = 2(|V(G)| - 1) - |E(G)|$.*

(ii) *(Catlin, Theorem 8 of [7]) G is simple and K_3-free.*

(iii) *(Catlin, Theorem 8 of [7]) $\delta(G) \leq 3$.*

(iv) *(Catlin et al., Theorem 1.3 of [9]) Either $G \in \{K_1, K_2\} \cup \{K_{2,t} : t \geq 1\}$ or $F(G) \geq 3$ and $|E(G)| \leq 2|V(G)| - 5$.*