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On the number of 7-cycles in regular n-tournaments

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Dedicated to Professor Domingos Moreira Cardoso for whom applying spectral graph theory to combinatorics is also his life-work

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ABSTRACT

For a regular tournament T of odd order n, let $c_m(T)$ be the number of cycles of length m in T. It is well known according to U. Colombo (1964) that $c_4(T) \leq c_4(RLT_n)$, where RLT_n is the unique regular locally transitive tournament of order n. In turn, in 1968, A. Kotzig proved that $c_4(DR_n) \leq c_4(T)$, where DR_n is a doubly-regular tournament of order n. However, the spectral tools allow us to simply show that the converse inequality $c_5(RLT_n) \leq c_5(T) \leq c_5(DR_n)$ holds. Recently we have proved that $c_6(T) \leq c_6(DR_n)$ and conjectured that $c_6(RLT_n) \leq c_6(T)$. For these values of m, the same results can be also formulated for the trace $tr_m(T)$ of the mth power of the adjacency matrix of T. (We consider this quantity here because it equals the number of closed walks of length m in T.) In the present paper, we determine $c_7(DR_n)$ and $c_7(RLT_n)$. Comparing $c_7(DR_n)$ with $c_7(RLT_n)$ yields the inequality $c_7(RLT_n) < c_7(DR_n)$, while $tr_7(DR_n) < tr_7(RLT_n)$ for $n \geq 7$. We also present some additional arguments which make it possible to suggest that for each odd $n \geq 9$, the two-sided bounds $c_7(RLT_n) \leq c_7(T) \leq c_7(DR_n)$ and $tr_7(DR_n) \leq tr_7(T) \leq tr_7(RLT_n)$ hold.

1. Introduction

A *tournament* T of order n is an orientation of the complete graph K_n. If a pair (i, j) is an arc in T, we say that the vertex i dominates the vertex j and write $i \to j$. For two vertex-sets I and J, we also write $I \Rightarrow J$ if every vertex of I dominates every vertex in J. The *in-set* $N^-(i)$ is the set of vertices dominating i in T. In turn, the *out-set* $N^+(i)$ is the set of vertices dominated by i in T. Obviously, $N^-(i) \Rightarrow i \Rightarrow N^+(i)$. Let $\delta_i^+ = |N^+(i)|$ and $\delta_i^- = |N^-(i)|$. The quantities δ_i^+ and δ_i^- are called the *out-degree* and *in-degree* of the vertex i, respectively. More generally, δ_i^+ and δ_i^- are the *semi-degrees* of the vertex i.

A tournament is *transitive* if $i \to k$ and $k \to j$ implies that $i \to j$. By L. Redei's theorem, any tournament of order n admits at least one hamiltonian path, say, $1, \ldots, n$. For the transitive case, we have $i \to j$ if $j \geq i + 1$. Obviously, this rule uniquely determines a tournament which we denote by $TT_n(1, \ldots, n)$. So, for given n, there exists exactly one transitive tournament TT_n of order n. Obviously, it is also *acyclic*, i.e. it admits no cycles, at all.

A tournament is *strongly connected* (or, simply, *strong*) if for any two distinct vertices i and j, there is a path from i to j. Let $s_m(T)$ be the number of strong subtournaments of order m in T. Since the subtournament induced by the union of i and a subset of $N^+(i)$ cannot be strongly connected, for $m \geq 3$, the following inequality (obtained first in [3])

$$s_m(T) \leq \binom{n}{m} - \sum_{i=1}^{n} \binom{\delta_i^+}{m-1} \tag{1}$$

holds with equality if every subtournament of order m in T is either strong or transitive. In particular, this condition is satisfied if T is *locally transitive*, i.e. the out-set and in-set of each vertex of T induce transitive tournaments.

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Note that $\sum_{i=1}^{n} \delta_i^+ = n(n-1)/2$. A well-known combinatorial result states that for a given sum $\sum_{i=1}^{n} \delta_i^+$, the binomial sum $\sum_{i=1}^{n} \binom{\delta_i^+}{m-1}$ is minimum and hence, the right-hand side of inequality (1) is a maximum when the out-degrees δ_i^+ are as nearly equal as possible (see [3]). That is, if n is odd, each δ_i^+ equals $\frac{n-1}{2}$ and hence, T is *regular*; if n is even, half the out-degrees are $\frac{n}{2}$ and the others are $\frac{n}{2}-1$, i.e. T is *near regular*. For both cases, the in-degrees take the same values as the out-degrees. So, if n is odd, we can say that a regular tournament of order n has semi-degree δ equal to $\frac{n-1}{2}$.

Denote by $\mathcal{T}_n$ the class of all tournaments of order n. For arbitrary $m \geq 3$, the arguments presented above imply that the maximum of $s_m(T)$ in the class $\mathcal{T}_n$, where n is odd, is attained at a *regular locally transitive* tournament of order n (see [3]). In Section 3, we show (after V. Dugat) that the regularity and local transitivity conditions uniquely determine T for each odd n. Such a tournament was first introduced in [10] and is also often called the *highly-regular* tournament, the regular *domination orientable* tournament, and even the *cyclonic* tournament (or, simply, *cyclone*). In this paper, we denote it by RLT_n.

Note that the tournament RLT_n is *rotational*, i.e. it can be represented as a tournament $R_n(S)$ on the ring $\mathbb{Z}_n = \{0, \dots, n-1\}$ of residues modulo n for which a pair (i, j) is an arc if and only if $j - i \in S$, where S is a subset of $\{1, \dots, n-1\}$ and the subtraction is taken modulo n. For $R_n(S)$ to be a tournament, the subset S must satisfy the conditions $S \cap -S = \emptyset$ and $S \cup -S = \{1, \dots, n-1\}$. In particular, for the considered case, we have $S = \{1, \dots, \frac{n-1}{2}\}$.

The tournament RLT_n need not be a unique maximizer of $s_m(T)$ in the class $\mathcal{T}_n$. In particular, the maximum of $s_3(T)$ in the class $\mathcal{T}_n$ is attained at a regular or near regular tournament according as n is odd or even (see [10]). It is so because a tournament of order 3 is either the cyclic triple Δ or TT_3 and hence, for $m = 3$, the equality always holds in (1).

If $m = 4$, the equality need not hold in (1) for an arbitrary tournament T of order n. Nevertheless, a formula for $s_4(T)$ can be obtained because all tournaments of order 4 can be also easily described. They are TT_4, $\circ \Rightarrow \Delta$, $\circ \Leftarrow \Delta$, and $RLT_5 - v$. The structure of these tournaments allows one to show (see [3,13]) that

$$s_4(T) = \binom{n}{4} - \sum_{i=1}^{n} \left\{ \binom{\delta_i^+}{3} + \binom{\delta_i^-}{3} \right\} + \sum_{(i,j)} \binom{\delta_{ij}^+}{2},$$

where δ_{ij}^+ is the number of vertices dominated by the pair of vertices i and j (we assume that $\delta_{ii}^+ = \delta_i^+$) and the second sum is taken over the arc-set of T. It is clear that if $i \to j$, then δ_{ij}^+ is the out-degree of j in the subtournament induced by the out-set of the vertex i. Hence, for a regular tournament with semi-degree δ, we have

$$\sum_{(i,j)} \delta_{ij}^+ = \sum_{i=1}^{n} \sum_{j \leftarrow i} \delta_{ij}^+ = (2\delta + 1)\delta(\delta - 1)/2.$$

This implies that the binomial sum $\sum_{(i,j)} \binom{\delta_{ij}^+}{2}$ takes the minimum value in the class $\mathcal{R}_{2\delta+1}$ of all regular tournaments with semi-degree δ (and hence, order $2\delta + 1$) when the numbers δ_{ij}^+ are as nearly equal as possible. Thus, the minimum of the above expression for $s_4(T)$ in the class $\mathcal{R}_{2\delta+1}$ is attained iff the out-set (and, hence, by duality, also in-set) of each vertex of T induces a regular or near regular tournament of order δ according as δ is odd or even (see [1,13]).

If this condition holds, then the corresponding tournament is *doubly-regular* or *nearly-doubly-regular* and is denoted by DR_n or NDR_n according as $n \equiv 3 \bmod 4$ or $n \equiv 1 \bmod 4$. In Section 5, we present a well-known infinite series of doubly-regular tournaments. However, the problem of the existence of DR_n for each $n \equiv 3 \bmod 4$ is open up to now, while according to a common opinion, it exists for any possible order. The same can be also said about NDR_n, where, recall, $n \equiv 1 \bmod 4$.

Let $c_m(T)$ be the number of cycles of length m (or, merely, m-cycles) in T. Since for $m = 3, 4$, there exists exactly one strongly connected tournament of order m and it contains precisely one Hamiltonian cycle, we have $c_m(T) = s_m(T)$ and hence, we can apply the above-mentioned classical results on $s_m(T)$ to $c_m(T)$. For $m = 5$, there exist exactly 12 tournaments of order 5 and the number of Hamiltonian cycles in them varies between 0 and 3. By this reason, it is difficult to get a clear combinatorial formula for $c_5(T)$, while the authors of [11] have been able to express $c_5(T)$ as the sum of the values of some polynomial function of four variables n, δ_i^+, δ_j^+, and δ_{ij}^+ taken over all arcs (i, j) of T. (As we have seen above, such an expression for $c_m(T)$ also exists if $m = 3$ or $m = 4$, but it is not so in the case of $m = 6$ because two nearly-doubly-regular tournaments of the same order need not have equal numbers of 6-cycles.)

As J.W. Moon pointed out on page 298 [14], the problem of determining the maximum of $c_m(T)$ in the class $\mathcal{T}_n$ seems to be very difficult in general. It is open up to now for each $m \geq 5$. However, the problem can be easily settled in the class $\mathcal{R}_n$ if $m = 5$. Indeed, the spectral methods allowed us to show in [19] (see formula (20) therein) that for a regular tournament T of order n, we have

$$c_5(T) + 2c_4(T) = \frac{n(n-1)(n+1)(n-3)(n+3)}{160}. \tag{2}$$

This identity and the results obtained for the case $m = 4$ directly imply the two-sided bounds

$$c_5(RLT_n) \leq c_5(T) \leq c_5(DR_n) \quad \text{if } n \equiv 3 \bmod 4$$

and

$$c_5(RLT_n) \leq c_5(T) \leq c_5(NDR_n) \quad \text{if } n \equiv 1 \bmod 4.$$