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Kernelization of the 3-path vertex cover problem

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ABSTRACT

The 3-path vertex cover problem is an extension of the well-known vertex cover problem. It asks for a vertex set $S \subseteq V(G)$ of minimum cardinality such that $G - S$ only contains independent vertices and edges. In this paper we will present a polynomial algorithm which computes two disjoint sets T_1, T_2 of vertices of G such that (i) for any 3-path vertex cover S' in $G[T_2]$, $S' \cup T_1$ is a 3-path vertex cover in G, (ii) there exists a minimum 3-path vertex cover in G which contains T_1 and (iii) $|T_2| \le 6 \cdot \psi_3(G[T_2])$, where $\psi_3(G)$ is the cardinality of a minimum 3-path vertex cover and T_2 is the kernel of G.

1. Introduction

During the last years the k-path vertex cover problem (k-PVCP for short) has become more and more interesting in graph theory since it is applicable to many practical problems. While there exist only a few results on k-PVCP, the number of open questions arises expeditiously. Motivated by the problem of ensuring data integrity of communication in wireless sensor networks using the k-generalized Canvas scheme in [6], Brešar et al. introduce the k-PVCP in [2].

A vertex subset $S \subseteq V(G)$ is a k-path vertex cover of G if $G - S$ contains no (not necessarily induced) path of length $k - 1$ in G. It is minimum if there exists no k-path vertex cover of smaller cardinality. We denote by $\psi_k(G)$ the cardinality of a minimum k-path vertex cover.

For $k = 2$, the k-PVCP is the well-known vertex cover problem (VCP for short), which is known to be NP-hard. Moreover, in [2] it is shown that the computation of $\psi_k(G)$ is NP-hard for $k \ge 3$.

Although the k-PVCP is NP-hard, there exist some approximation algorithms for $k \le 3$. For example, using Nemhauser's and Trotter's result in [5], we have a factor-2 algorithm for $k = 2$. For larger k, it is widely unknown whether one can approximate the k-PVCP within a factor smaller than k. An exceptional case is $k = 3$, where two factor-2 algorithms are given by Tu and Zhou in [9] and [10].

For $k = 3$, one can find an approximation algorithm and some bounds for $\psi_3(G)$ in cubic graphs in [8], whereas [4] gives an exact algorithm to solve the 3-PVCP in time $\mathcal{O}^*(1.5171^n)$ for general graphs.

By Nemhauser's and Trotter's paper in 1975 [5], the question of finding the "hard part" of an NP-hard problem in a graph G arises. In that sense "hard part" means kernel, i.e. the remaining set of vertices after applying some polynomial reduction techniques. In [5], the authors deal with the VCP, i.e. 2-PVCP, and its kernel. Given d, the generalization of the VCP of Fellows et al. in [3] considers the problem of finding a vertex set of minimum cardinality whose removal from G yields a graph possessing vertices of degree at most d. Their result provides a polynomial algorithm computing a vertex set T such that the cardinality of an optimal solution is at most $|T|/(d^3 + 4d^2 + 6d + 4)$. It gives us a first kernelization algorithm for the 3-PVCP.

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On the one hand, we have two factor-2 approximation algorithms [9,10], which do not use kernelization techniques. On the other hand, we can compute a kernel $T \subseteq V(G)$ polynomially, such that $|T| \leq 15 \cdot \psi_3(G[T])$ (by [3] for $d = 1$). Since the range between 2 and 15 is really large, the aim of this paper is to provide a polynomial algorithm which computes a better kernel T for an arbitrary graph G, i.e. $|T| \leq 6 \cdot \psi_3(G[T])$.

We consider finite, simple and undirected graphs and use [1] for terminology and notation which are not defined here.

A vertex subset $S \subseteq V(G)$ is *independent* (*dissociative*) if $G[S]$ contains no P_2 (no P_3). A set of vertex disjoint P_3's is a P_3-*packing*. It is *maximal* if there exists no P_3 in G containing no vertex of a P_3 in the P_3-*packing*. For some maximal P_3-packing $\mathcal{P}$, the graph $G[V(G) \setminus V(\mathcal{P})]$ is the disjoint union of isolated vertices and isolated edges, i.e. $V(G) \setminus V(\mathcal{P})$ is a dissociative set in G. Let us denote by $\mathcal{P}_2(\mathcal{P})$ its set of isolated P_2's and by $\mathcal{P}_1(\mathcal{P})$ its set of isolated vertices. Furthermore, let us define $Q(\mathcal{P})$ as set of those vertices in $V(\mathcal{P})$ which have a neighbour in $V(\mathcal{P}_1(\mathcal{P})) \cup V(\mathcal{P}_2(\mathcal{P}))$. If P is a path in $\mathcal{P}$, then let us denote by $Q(P)$ the set of vertices in P which are in $Q(\mathcal{P})$. To simplify notation, let us say vertices in $V(\mathcal{P})$ are *black* and vertices in $V(\mathcal{P}_1(\mathcal{P})) \cup V(\mathcal{P}_2(\mathcal{P}))$ are *white* for some given $\mathcal{P}$. Moreover, all vertices in $Q(\mathcal{P})$ are called *contact* vertices. We define the white neighbourhood $N^w(u)$ of a black vertex u as the subset of white vertices which are either adjacent to u or have one white common neighbour with u. Additionally, let us define $N_1^w(u)$ and $N_2^w(u)$ as $N^w(u) \cap V(\mathcal{P}_1(\mathcal{P}))$ and $N^w(u) \cap V(\mathcal{P}_2(\mathcal{P}))$, respectively. Again to simplify notation, let us denote for some dissociative set D by $Q'(D)$ the set of all vertices $u \in N(D)$ for which every neighbour in D is not isolated in $G[D]$. To generalize our concepts, let us define by $f(T)$ the set $\bigcup_{u \in T} f(u) \setminus T$ for some function $f : V(G) \to 2^{V(G)}$ and some set $T \subseteq V(G)$.

2. Results

Our main objective is to provide a polynomial algorithm computing a kernel of the 3-path vertex cover problem in G. We need two important tools for it. First, we introduce the concept of a 3-path crown decomposition.

Definition. A 3-*path crown decomposition* (H, C, R) is a partition of the vertices of the graph G such that

(i) H (the header) separates C and R, i.e. there exist no edges between C and R,
(ii) C (the crown) is a dissociative set in G,
(iii) there exists a function $F : H \to \binom{C \cup H}{3}$ such that $\{G[F(u)] : u \in H\}$ is a P_3-packing in $G[H \cup C]$ of cardinality $|H|$ whose every path contains exactly one vertex of H.

Special cases of the 3-path crown decomposition are introduced by Prieto and Sloper in [7] and are known as *fat crown decomposition* and *double crown decomposition*. The first one requires the additional property that only end-vertices of the P_3's in the P_3-packing are elements of H while the second one considers C as an independent set in G.

The usefulness of the 3-path crown decomposition is presented in the next lemma.

Lemma 2.1. *A graph G that admits a* 3-*path crown decomposition (H, C, R) has a* 3-*path vertex cover of size at most c if and only if $G[R]$ has a* 3-*path vertex cover of size at most $c - |H|$.*

Proof. Since C is a dissociative set in G, $S' \cup H$ is a 3-path vertex cover of size $|S'| + |H|$ in G if $S' \subseteq R$ is a 3-path vertex cover in $G[R]$.

Let S be a 3-path vertex cover in G. Assume $H \cap (V(G) \setminus S) \neq \emptyset$. Due to the definition of a 3-path crown decomposition, we have a P_3-packing in $G[H \cup C]$ of cardinality $|H|$ where each P_3 has exactly one vertex in H. Since these P_3 are covered by H, it follows $|C \cap S| \geq |H \cap (V(G) \setminus S)|$. This inequality implies that $S \setminus (H \cup C)$ is a 3-path vertex cover of size at most $|S| - |H|$. □

According to the above lemma, $\psi_3(G) = \psi_3(G[R]) + |H|$ follows easily by deleting C and H of a 3-path crown decomposition. It indicates the importance of the lemma. Our aim is to provide a polynomial kernelization algorithm by using the concept of 3-path crown decomposition. The computation of the latter one can be divided into two steps.

The first one considers the fat crown decomposition.

Lemma 2.2 (*Prieto and Sloper [7]*)**.** *Let G be a graph and $\mathcal{J}$ be a collection of independent P_2's such that $|\mathcal{J}| \geq |N(V(\mathcal{J}))|$. Then we can find a fat crown decomposition (H, C, R) where $C \subseteq V(\mathcal{J})$ and $H \subseteq N(V(\mathcal{J}))$ in linear time.*

Let G be a graph and D be a dissociative set. To find a 3-path crown decomposition, we contract all edges in D and obtain a graph G^*. By the above lemma, for some given dissociative set D in G either one can find a fat crown decomposition in linear time, which perhaps is a 3-path crown decomposition, or the number of contracted edges is bounded from above by $|N(D)|$. It helps us for the second step, which basically uses the property that we obtain an independent set by contracting all edges in D.

Lemma 2.3. *For a graph G and a dissociative set D, let G^* be the graph constructed by edge contraction in D and adding an additional vertex u', which is only adjacent to u, for every vertex $u \in Q'(D)$. Furthermore, let us denote by D^* the set $V(G^*) \setminus (V(G) \setminus D)$. If (H, C^*, R) is a double crown decomposition in G^* such that $C^* \subseteq D^*$ and $H \subseteq N(D^*)$, then $(H, V(G) \setminus (H \cup R), R)$ is a* 3-*path crown decomposition in G such that $V(G) \setminus (H \cup R) \subseteq D$ and $H \subseteq N(D)$.*