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Vertex magic total labelings of 2-regular graphs

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ABSTRACT

A *vertex magic total (VMT) labeling* of a graph $G = (V, E)$ is a bijection from the set of vertices and edges to the set of integers defined by $\lambda : V \cup E \to \{1, 2, \ldots, |V|+|E|\}$ so that for every $x \in V$, $w(x) = \lambda(x)+\sum_{xy\in E}\lambda(xy) = k$, for some integer k. A VMT labeling is said to be a *super* VMT labeling if the vertices are labeled with the smallest possible integers, $1, 2, \ldots, |V|$. In this paper we introduce a new method to expand some known VMT labelings of 2-regular graphs.

1. Introduction

Sedláček introduced the notion of a magic labeling in 1963 [7]. He defined a magic labeling of a graph G as a bijection f from E to a set of positive integers such that f holds the conditions:

(a) $f(e_i) \neq f(e_j)$ for all $e_i, e_j \in E$,
(b) $\sum_{e\in N_E(x)} f(e)$ is constant for every $x \in V$, where $N_E(x)$ is the set of edges incident to x.

Kotzig and Rosa [3] and MacDougall et al. [4] defined two other magic labelings: the *edge-magic total (EMT) labeling* and the *vertex-magic total (VMT) labeling*, respectively.

Let G be a simple graph with vertex set V and edge set E. A *total labeling* of G is a bijection $f : V \cup E \to \{1, 2, \ldots, |V|\cup|E|\}$. If $x, y \in V$ and if $e = xy \in E$, then the weight $w_f^t(e)$ of e is given by $w_f^t(e) = f(x) + f(y) + f(e)$. A total labeling f is said to be an *edge-magic total labeling* if the weight of each edge is equal to the same constant k. The weight $w_f^t(x)$ of a vertex x is defined as $w_f^t(x) = f(x) + \sum_{xy\in E} f(xy)$ where the sum is over all edges xy incident with the vertex x. A total labeling f is said to be a *vertex-magic total labeling* (VMT labeling) if the weight of each vertex is equal to the same constant k, called the *magic constant* of the VMT labeling. A VMT labeling is said to be a *super* VMT labeling if the vertices are labeled with the smallest possible integers, $1, 2, \ldots, |V|$.

In [2], Holden, McQuillan, and McQuillan posed the following conjecture.

Conjecture 1. *A 2-regular graph of odd order possesses a super VMT labeling if and only if it is not one of* $C_4 \cup C_3$, $C_4 \cup 3C_3$ *or* $C_5 \cup 2C_3$.

In [6], D. McQuillan introduced a technique for constructing magic labelings of 2-regular graphs and proved the following result, which contributes significantly to Conjecture 1.

We denote by $H_1 \cup H_2 \cup \cdots \cup H_n$ the usual disjoint union of graphs.

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Theorem 1.1 ([6]). *Let $G = C_{h_1} \cup C_{h_2} \cup \cdots \cup C_{h_l}$. Let $I = \{1, 2, \ldots, l\}$ and J be any subset of I. Let*

$$G_J = \left(\bigcup_{i \in J} nC_{h_i} \right) \cup \left(\bigcup_{i \in I-J} C_{nh_i} \right),$$

where n is an odd number with $n = 2m+1$. If G has a VMT labeling with magic constant k, then G_J has VMT labelings with magic constants $k_1 = 6m(h_1 + h_2 + \cdots + h_l) + k$ and $k_2 = nk - 3m$.

In [3], Kotzig and Rosa proved a theorem about preserving EMT labelings for odd number of copies of certain graphs.

Theorem 1.2. *If G is a 3-colorable edge-magic graph and H is the union of $t = 2s + 1$ disjoint copies of G, then H is edge magic.*

It is easy to see that EMT and VMT labelings of 2-regular graphs are equivalent. In this paper we use Theorem 1.2 to expand VMT labeling as we multiply the number of cycles.

In Section 2 we introduce a new method for finding VMT labelings of 2-regular graphs with expanded cycles from known VMT labelings of certain 2-regular graphs. In Section 3 we list known and new results on VMT labelings for some families of graphs, with some examples describing how the method works. In Section 4 we present VMT labelings for some families of 2-regular graphs where no such labelings were known so far. These results were presented in an MS Thesis of the third author [8].

2. Extending lengths of cycles

In this section we introduce a method that can be applied to 2-regular graphs. This method preserves the VMT (SVMT) properties as we extend the length of a 2-regular graph by a factor of an odd number.

The only 2-regular graphs are cycles and unions of cycles. In this section we show how our method can be used to preserve the properties of vertex magic total labelings as we extend the length of each cycle of a disjoint union mC_n to $mC_{n(2r+1)}$, where r is a positive integer.

In [5] Marr and Wallis give a definition of a Kotzig array as a $d \times m$ grid, each row being a permutation of $\{0, 1, \ldots, m-1\}$ and each column having the same sum. The Kotzig array used in this paper is the $3 \times (2r+1)$ Kotzig array that is given as an example in [5]:

$$\begin{bmatrix} 0 & 1 & \ldots & r & r+1 & r+2 & \ldots & 2r \\ 2r & 2r-2 & \ldots & 0 & 2r-1 & 2r-3 & \ldots & 1 \\ r & r+1 & \ldots & 2r & 0 & 1 & \ldots & r-1 \end{bmatrix}.$$

Let κ be a $d \times m$ grid, which is obtained by adding 1 to every element and switching the second and third rows of a Kotzig array.

$$\kappa = \begin{bmatrix} 1 & 2 & \ldots & r+1 & r+2 & \ldots & 2r & 2r+1 \\ r+1 & r+2 & \ldots & 2r+1 & 1 & \ldots & r-1 & r \\ 2r+1 & 2r & \ldots & 1 & 2r-1 & \ldots & 4 & 2 \end{bmatrix}.$$

If we write the first two rows of κ as a permutation cycle τ, we have:

$$\tau = (1, r+1, 2r+1, r, 2r, \ldots, 3, r+3, 2, r+2).$$

The difference between any two consecutive elements in τ is equal to r taken modulo $(2r+1)$. Note that τ is a $(2r+1)$-cycle. Since $(2r+1)$ is an odd number for every nonnegative integer r, then $\gcd(2, 2r+1) = 1$, and hence the permutation τ^2 is also a $(2r+1)$-cycle. This fact plays an important role in preserving the magic properties of our VMT and SVMT labelings as we extend the length of cycles in a 2-regular graph.

Let κ' be a modified κ, where we switch the first and second row of κ:

$$\kappa' = \begin{bmatrix} r+1 & r+2 & \ldots & 2r+1 & 1 & \ldots & r-1 & r \\ 1 & 2 & \ldots & r+1 & r+2 & \ldots & 2r & 2r+1 \\ 2r+1 & 2r & \ldots & 1 & 2r-1 & \ldots & 4 & 2 \end{bmatrix}.$$

Obviously, if we write the first two rows of κ' as a permutation cycle, we have τ^{-1}.

Theorem 2.1. *Let G be a 2-regular graph that has a VMT labeling μ. Let G' be a 2-regular graph obtained by extending the length of each component of G by an odd factor. Then there exists a VMT labeling for G' that can be obtained by modifying the VMT labeling of G.*

Proof. Let μ be a VMT labeling of any 2-regular graph G. For every vertex and edge of G, let λ be the labeling obtained by decreasing the original label by 1, that is, let $\lambda(v) = \mu(v) - 1$ and $\lambda(e) = \mu(e) - 1$.