Discrete Mathematics 340 (2017) 3125–3139

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Automorphism groups of Cayley graphs generated by block transpositions and regular Cayley maps

Annachiara Korchmaros [a], István Kovács [b,*]

[a] School of Mathematics Georgia Institute of Technology, 686 Cherry Street Atlanta, GA 30332-0160, USA
[b] IAM, University of Primorska, Muzejski trg 2, SI-6000 Koper, Slovenia

ARTICLE INFO

Article history:
Received 23 October 2015
Received in revised form 13 May 2016
Accepted 15 June 2016
Available online 4 August 2016

Keywords:
Cayley graph
Symmetric group
Block transposition
Graph automorphism
Cayley map
Regular map

ABSTRACT

This paper deals with the Cayley graph $\mathrm{Cay}(\mathrm{Sym}_n, T_n)$, where the generating set consists of all block transpositions. A motivation for the study of these particular Cayley graphs comes from current research in Bioinformatics. As the main result, we prove that $\mathrm{Aut}(\mathrm{Cay}(\mathrm{Sym}_n, T_n))$ is the product of the left translation group and a dihedral group D_{n+1} of order $2(n+1)$. The proof uses several properties of the subgraph Γ of $\mathrm{Cay}(\mathrm{Sym}_n, T_n)$ induced by the set T_n. In particular, Γ is a $2(n-2)$-regular graph whose automorphism group is D_{n+1}, Γ has as many as $n+1$ maximal cliques of size 2, and its subgraph $\Gamma(V)$ whose vertices are those in these cliques is a 3-regular, Hamiltonian, and vertex-transitive graph. A relation of the unique cyclic subgroup of D_{n+1} of order $n+1$ with regular Cayley maps on Sym_n is also discussed. It is shown that the product of the left translation group and the latter group can be obtained as the automorphism group of a non-t-balanced regular Cayley map on Sym_n.

1. Introduction

Block transpositions are well-known sorting operations with relevant applications in Bioinformatics, see [11]. They act on a string by removing a block of consecutive entries and inserting it somewhere else. In terms of the symmetric group Sym_n of degree n, the strings are identified with the permutations on $[n] = \{1, 2, \ldots, n\}$, and block transpositions are defined as follows. For any three integers i, j, k with $0 \le i < j < k \le n$, the *block transposition* $\sigma(i, j, k)$ with cut points (i, j, k) turns the permutation $\pi = [\pi_1 \ldots \pi_n]$ into the permutation $\pi' = [\pi_1 \cdots \pi_i\ \pi_{j+1} \cdots \pi_k\ \pi_{i+1} \cdots \pi_j\ \pi_{k+1} \cdots \pi_n]$. This action of $\sigma(i, j, k)$ on π can also be expressed as the composition $\pi' = \pi \circ \sigma(i, j, k)$. The set T_n of all block transpositions has size $(n+1)n(n-1)/6$ and is an inverse closed generating set of Sym_n. The arising Cayley graph $\mathrm{Cay}(\mathrm{Sym}_n, T_n)$ is a very useful tool since "sorting a permutation by block transpositions" is equivalent to finding shortest paths between vertices in $\mathrm{Cay}(\mathrm{Sym}_n, T_n)$, see [4,8,9,11,20].

Although the definition of a block transposition arose from a practical need, the Cayley graph $\mathrm{Cay}(\mathrm{Sym}_n, T_n)$ also presents some interesting theoretical features (cf. [16]); and the most remarkable one is perhaps the existence of automorphisms other than left translations. We remark that the automorphism groups of certain Cayley graphs on symmetric groups have attracted some attention recently, see [7,10,12].

In this paper, we focus on the automorphisms of the Cayley graph $\mathrm{Cay}(\mathrm{Sym}_n, T_n)$, and denote its full automorphism group by $\mathrm{Aut}(\mathrm{Cay}(\mathrm{Sym}_n, T_n))$. In particular, we introduce a subgroup of automorphisms contained in the vertex stabilizer of the

* Corresponding author.
E-mail addresses: akorchmaros3@gatech.edu (A. Korchmaros), istvan.kovacs@upr.si (I. Kovács).

http://dx.doi.org/10.1016/j.disc.2016.06.014

identity permutation ι, which is isomorphic to the dihedral group of order $2(n+1)$. We denote this dihedral group by D_{n+1}, and since it arises from the toric equivalence in Sym_n and the reverse permutation, we name it the *toric-reverse group*.

In addition, the *block transposition graph*, that is the subgraph Γ of $\mathrm{Cay}(\mathrm{Sym}_n, T_n)$ induced by the vertex set T_n has especially nice properties. As we show in this paper, Γ is a $2(n-2)$-regular graph whose automorphism group coincides with the permutation group induced by toric-reverse group D_{n+1}. Clearly, as every automorphism in D_{n+1} fixes ι, it also preserves the neighborhood of ι in $\mathrm{Cay}(\mathrm{Sym}_n, T_n)$, which is the set T_n. Therefore, D_{n+1} induces an automorphism group of Γ. We show that the latter group is, in fact, equal to $\mathrm{Aut}(\Gamma)$, the full automorphism group of Γ.

Theorem 1. *The automorphism group of Γ is equal to the permutation group induced by toric-reverse group* D_{n+1}.

The key part of the proof of Theorem 1 is a careful analysis of the maximal cliques of Γ of size 2. We show that Γ has precisely $n+1$ such cliques and also look inside the subgraph $\Gamma(V)$ of Γ induced by the set V whose vertices are the $2(n+1)$ vertices of these cliques. We prove that $\Gamma(V)$ is 3-regular, and D_{n+1} induces an automorphism group of $\Gamma(V)$ acting transitively (and hence regularly) on V. As a curiosity, we also observe that $\Gamma(V)$ is Hamiltonian. This agrees with a result of Alspach and Zhang [2] and confirms the Lovász conjecture [18] for $\Gamma(V)$.

Then, we turn to the automorphism group $\mathrm{Aut}(\mathrm{Cay}(\mathrm{Sym}_n, T_n))$, and using Theorem 1, we show that any automorphism fixing ι belongs to D_{n+1}. Therefore, $\mathrm{Aut}(\mathrm{Cay}(\mathrm{Sym}_n, T_n))$ will be completely determined.

Theorem 2. *The automorphism group of* $\mathrm{Cay}(\mathrm{Sym}_n, T_n)$ *is the product of the left translation group and the toric-reverse group* D_{n+1}.

Finally, we also discuss a relation of the *toric group*$\bar{\mathsf{F}}$, the unique cyclic subgroup of D_{n+1} of order $n+1$, with regular Cayley maps defined on the symmetric group Sym_n. We show that if $n \geq 5$, then the product of the left translation group and the toric group $\bar{\mathsf{F}}$ can be characterized as the automorphism group of a non-t-balanced regular Cayley map on Sym_n (see Theorem 4). This follows from the observation that any element in $\bar{\mathsf{F}}$ is a skew-morphism of the group Sym_n in the sense of [15].

2. Background on block transpositions

In this paper, all groups are finite, and all graphs are finite and simple. For basic facts on Cayley graphs and combinatorial properties of permutations the reader is referred to [1,3,4].

Throughout the paper, n denotes a positive integer. In our investigation cases $n \leq 3$ are trivial while case $n = 4$ presents some results different from the general case.

For a set X of size n, Sym_X stands for the set of all permutations on X. For the sake of simplicity, $[n] = \{1, 2, \ldots, n\}$ is usually taken for X. As it is customary in the literature on block transpositions, we mostly adopt the functional notation for permutations: If $\pi \in \mathrm{Sym}_n$, then $\pi = [\pi_1\pi_2 \cdots \pi_n]$ with $\pi(t) = \pi_t$ for every $t \in [n]$, and if $\pi, \rho \in \mathrm{Sym}_n$ then $\tau = \pi \circ \rho$ is the permutation defined by $\tau(t) = \pi(\rho(t))$ for every $t \in [n]$. The *reverse permutation* is $w = [n\, n-1 \cdots 1]$, and $\iota = [1\,2 \cdots n]$ is the *identity permutation*.

For any three integers, named *cut points*, (i, j, k) with $0 \leq i < j < k \leq n$, the *block transposition* (transposition, see [13]) $\sigma(i, j, k)$ is defined to be the function on $[n]$:

$$\sigma(i,j,k)_t = \begin{cases} t, & 1 \leq t \leq i, \quad k+1 \leq t \leq n, \\ t+j-i, & i+1 \leq t \leq k-j+i, \\ t+j-k, & k-j+i+1 \leq t \leq k. \end{cases} \tag{1}$$

This shows that $\sigma(i, j, k)_{t+1} = \sigma(i, j, k)_t + 1$ in the intervals:

$$[1, i], \qquad [i+1, k-j+i], \qquad [k-j+i+1, k], \qquad [k+1, n], \tag{2}$$

where

$$\begin{gathered} \sigma(i,j,k)_i = i; \qquad \sigma(i,j,k)_{i+1} = j+1; \qquad \sigma(i,j,k)_{k-j+i} = k; \\ \sigma(i,j,k)_{k-j+i+1} = i+1; \qquad \sigma(i,j,k)_k = j; \qquad \sigma(i,j,k)_{k+1} = k+1. \end{gathered} \tag{3}$$

Actually, $\sigma(i, j, k)$ can also be represented as the permutation

$$\sigma(i,j,k) = \begin{cases} [1 \cdots i\; j+1 \cdots k\; i+1 \cdots j\; k+1 \cdots n], & 1 \leq i,\; k < n, \\ [j+1 \cdots k\; 1 \cdots j\; k+1 \cdots n], & i = 0,\; k < n, \\ [1 \cdots i\; j+1 \cdots n\; i+1 \cdots j], & 1 \leq i,\; k = n, \\ [j+1 \cdots n\; 1 \cdots j], & i = 0,\; k = n \end{cases} \tag{4}$$

such that the action of $\sigma(i, j, k)$ on π is defined as the product

$$\pi \circ \sigma(i,j,k) = [\pi_1 \cdots \pi_i\; \pi_{j+1} \cdots \pi_k\; \pi_{i+1} \cdots \pi_j\; \pi_{k+1} \cdots \pi_n].$$