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# Location-domination in line graphs

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ABSTRACT

A set $D$ of vertices of a graph $G$ is locating if every two distinct vertices outside $D$ have distinct neighbors in $D$; that is, for distinct vertices $u$ and $v$ outside $D$, $N(u) \cap D \neq N(v) \cap D$, where $N(u)$ denotes the open neighborhood of $u$. If $D$ is also a dominating set (total dominating set), it is called a locating-dominating set (respectively, locating-total dominating set) of $G$. A graph $G$ is twin-free if every two distinct vertices of $G$ have distinct open and closed neighborhoods. It is conjectured (Garijo et al., 2014 [15]) and (Foucaud and Henning, 2016 [12]) respectively, that any twin-free graph $G$ without isolated vertices has a locating-dominating set of size at most one-half its order and a locating-total dominating set of size at most two-thirds its order. In this paper, we prove these two conjectures for the class of line graphs. Both bounds are tight for this class, in the sense that there are infinitely many connected line graphs for which equality holds in the bounds.



## 1. Introduction

In this paper, we prove two recent conjectures on locating-dominating sets and locating-total dominating sets in graphs for the class of line graphs. In order to state these conjectures, we define the necessary graph theory terminology that we shall use. A *dominating set* in a graph $G$ is a set $D$ of vertices of $G$ such that every vertex outside $D$ is adjacent to a vertex in $D$, while a *total dominating set*, abbreviated TD-set, of $G$ is a dominating set with the additional property that every vertex inside $D$ is also adjacent to a vertex in $D$. The *domination number*, $\gamma(G)$, and the *total domination number* of $G$, denoted by $\gamma_t(G)$, is the minimum cardinality of a dominating set and a TD-set, respectively, in $G$. The literature on the subject of domination parameters in graphs up to the year 1997 has been surveyed and detailed in the two books [17,16], and a recent book on total dominating sets is also available [21].

A *neighbor* of a vertex $v$ in $G$ is a vertex adjacent to $v$ in $G$, while the *open neighborhood of* $v$ is the set of all neighbors of $v$ in $G$. The *closed neighborhood of* $v$ consists of all neighbors of $v$ together with the vertex $v$. A graph is *twin-free* if every two distinct vertices have distinct open and closed neighborhoods.

Among the existing variations of (total) domination, the one of *location-domination* and *location-total domination* are widely studied. A set $D$ of vertices *locates* a vertex $v \notin D$ if the neighborhood of $v$ within $D$ is unique among all vertices in $V(G) \setminus D$. A *locating-dominating set* is a dominating set $D$ that locates all the vertices in $V(G) \setminus D$, and the *location-domination number* of $G$, denoted $\gamma_L(G)$, is the minimum cardinality of a locating-dominating set in $G$. A *locating-total dominating set*, abbreviated LTD-set, is a TD-set $D$ that locates all the vertices, and the *location-total domination number* of $G$, denoted $\gamma_t^L(G)$, is the minimum cardinality of a LTD-set in $G$. The concept of a locating-dominating set was introduced and first studied by

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Slater [26,27] (see also [9,10,14,25,28]), and the additional condition that the locating-dominating set be a total dominating set was first considered in [18] (see also [1–3,5–7,19,20]).

A classic result in domination theory due to Ore [24] states that every graph without isolated vertices has a dominating set of cardinality at most one-half its order. This bound is tight and the extremal examples have been classified, see [23]. As observed in [14], while there are many graphs (without isolated vertices) which have location-domination number much larger than one-half their order, the only such graphs that are known contain many twins. For example, for the complete graph $K_n$ of order $n$, we have $\gamma_L(K_n) = n-1$ for all $n \geq 3$. It was therefore recently conjectured by Garijo et al. [15] that for sufficiently large values of the order and in the absence of twins and multiple components, the classic bound of one-half the order for the domination number also holds for the location-domination number.

**Conjecture 1** (*Garijo, González, Márquez [15]*). *There exists an integer $n_1$ such that for any $n \geq n_1$, the maximum value of the location-domination number of a connected twin-free graph of order $n$ is $\lfloor \frac{n}{2} \rfloor$.*

We proposed in [13,14] the following strengthening of Conjecture 1.[1]

**Conjecture 2** (*Foucaud, Henning, Löwenstein and Sasse [13,14]*). *Every twin-free graph $G$ of order $n$ without isolated vertices satisfies $\gamma_L(G) \leq \frac{n}{2}$.*

Garijo et al. [15] proved that for any $n \geq 14$, the maximum value of the location-domination number of a connected twin-free graph is at least $\lfloor \frac{n}{2} \rfloor$. Thus, together with this fact, the statement of Conjecture 2 implies the statement of Conjecture 1.

A classic result in total domination theory due to Cockayne et al. [8] states that every graph with components of order at least 3 has a TD-set of cardinality at most two-thirds its order. This bound is tight and the extremal examples have been classified, see [4]. As observed in [12], while there are many such graphs which have location-total domination number much larger than two-thirds their order, the only such graphs that are known contain many twins. For example, for the star $K_{1,n-1}$ of order $n$, we have $\gamma_t^L(K_{1,n-1}) = n-1$ for all $n \geq 3$. The authors in [12] conjectured that in the absence of twins, the classic bound of two-thirds the order for the total domination number also holds for the locating-total domination number.

**Conjecture 3** (*Foucaud and Henning [12]*). *Every twin-free graph $G$ of order $n$ without isolated vertices satisfies $\gamma_t^L(G) \leq \frac{2}{3}n$.*

In this paper, we focus on the class of line graphs. We prove the two conjectures for this class, and discuss extremal examples. The key for this study is to define *edge-locating-(total) dominating sets* (similar to edge-dominating sets) and to study this concept in general graphs.

**Definitions and Notation**. For notation and graph theory terminology, we in general follow [17]. Specifically, let $G$ be a graph with vertex set $V(G)$, edge set $E(G)$ and with no isolated vertex. The *open neighborhood* of a vertex $v \in V(G)$ is $N_G(v) = \{u \in V \mid uv \in E(G)\}$ and its *closed neighborhood* is the set $N_G[v] = N_G(v) \cup \{v\}$. The degree of $v$ is $d_G(v) = |N_G(v)|$. For a set $S \subseteq V(G)$, its *open neighborhood* is the set $N_G(S) = \bigcup_{v \in S} N_G(v)$, and its *closed neighborhood* is the set $N_G[S] = N_G(S) \cup S$. If the graph $G$ is clear from the context, we simply write $V$, $E$, $N(v)$, $N[v]$, $N(S)$, $N[S]$ and $d(v)$ rather than $V(G)$, $E(G)$, $N_G(v)$, $N_G[v]$, $N_G(S)$, $N_G[S]$ and $d_G(v)$, respectively.

Given a set $S$ of edges, we will denote by $G - S$ the subgraph obtained from $G$ by deleting all edges of $S$. For a set $S$ of vertices, $G - S$ is the graph obtained from $G$ by removing all vertices of $S$ and removing all edges incident with vertices of $S$. The subgraph induced by a set $S$ of vertices (respectively, edges) in $G$ is denoted by $G[S]$. A *cycle* on $n$ vertices is denoted by $C_n$ and a *path* on $n$ vertices by $P_n$. A complete graph on four vertices minus one edge is called a *diamond*. The *girth* of $G$ is the length of a shortest cycle in $G$. A *leaf* of $G$ is a vertex of degree 1 in $G$, while a *pendant edge* of $G$ is an edge of $G$ with at least one of its ends a leaf.

A *rooted tree* distinguishes one vertex $r$ called the *root*. For each vertex $v \neq r$ of $T$, the *parent* of $v$ is the neighbor of $v$ on the unique $(r, v)$-path, while a *child* of $v$ is any other neighbor of $v$. A *descendant* of $v$ is a vertex $u \neq v$ such that the unique $(r, u)$-path contains $v$. Let $D(v)$ denote the set of descendants of $v$, and let $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at $v$ is the subtree of $T$ induced by $D[v]$, and is denoted by $T_v$.

A set $D$ is a dominating set of $G$ if $N[v] \cap D \neq \emptyset$ for every vertex $v$ in $G$, or, equivalently, $N[D] = V(G)$. A set $D$ is a total dominating set of $G$ if $N(v) \cap D \neq \emptyset$ for every vertex $v$ in $G$, or, equivalently, $N(D) = V(G)$. Two distinct vertices $u$ and $v$ in $V(G) \setminus D$ are *located* by $D$ if they have distinct neighbors in $D$; that is, $N(u) \cap D \neq N(v) \cap D$. If a vertex $u \in V(G) \setminus D$ is located from every other vertex in $V(G) \setminus D$, we simply say that $u$ is *located* by $D$.

A set $S$ is a *locating set* of $G$ if every two distinct vertices outside $S$ are located by $S$. In particular, if $S$ is both a dominating set and a locating set, then $S$ is a locating-dominating set. Further, if $S$ is both a total dominating set and a locating set, then $S$ is a *locating-total dominating set* (where $S$ is a *total dominating set* of $G$ if every vertex of $G$ is adjacent to some vertex in $S$). We remark that the only difference between a locating set and a locating-dominating set in $G$ is that a locating set might have a unique non-dominated vertex.

An *independent set* in $G$ is a set of vertices no two of which are adjacent. The *independence number* of $G$, denoted $\alpha(G)$, is the maximum cardinality of an independent set of vertices in $G$. The complement of an independent set in $G$ is a *vertex cover* in $G$. Thus if $S$ is a vertex cover in $G$, then every edge of $G$ is incident with at least one vertex in $S$.

[1] Note that in [14], we mistakenly attributed Conjecture 2 to the authors of [15]. We discuss this in more detail in [13].