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Note

A note on the colorful fractional Helly theorem

Minki Kim

Department of Mathematical Sciences, KAIST, Daejeon, South Korea

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ABSTRACT

Helly's theorem is a classical result concerning the intersection patterns of convex sets in $\mathbb{R}^d$. Two important generalizations are the colorful version and the fractional version. Recently, Bárány et al. combined the two, obtaining a colorful fractional Helly theorem. In this paper, we give an improved version of their result.

1. Introduction

Helly's theorem is one of the most well-known and fundamental results in combinatorial geometry, which has various generalizations and applications. It was first proved by Helly [12] in 1913, but his proof was not published until 1923, after alternative proofs by Radon [17] and König [15]. We recommend the survey paper by Amenta, Loera, and Soberón [4] for an overview of previous results and open problems related to Helly's theorem. Recall that a family is *intersecting* if the intersection of all members is non-empty. The following is the original version of Helly's theorem.

Theorem 1.1 (*Helly's Theorem, Helly [12]*). *Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$ with $|\mathcal{F}| \geq d+1$. Suppose every $(d+1)$-tuple of $\mathcal{F}$ is intersecting. Then the whole family $\mathcal{F}$ is intersecting.*

The following variant of Helly's theorem was found by Lovász, whose proof appeared first in a paper by Bárány [5]. Note that the original theorem by Helly is obtained by setting $\mathcal{F}_1 = \mathcal{F}_2 = \cdots = \mathcal{F}_{d+1}$.

Theorem 1.2 (*Colorful Helly Theorem, Lovász [5]*). *Let $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{d+1}$ be finite, non-empty families (color classes) of convex sets in $\mathbb{R}^d$ such that every colorful $(d+1)$-tuple is intersecting. Then, for some $1 \leq i \leq d+1$, the whole family $\mathcal{F}_i$ is intersecting.*

One way to generalize Helly's theorem is by weakening the assumption: not necessarily all but only a positive fraction of $(d+1)$-tuples are intersecting. The following theorem shows how the conclusion changes.

Theorem 1.3 (*Fractional Helly Theorem, Katchalski and Liu [14]*). *For every $\alpha \in (0,1]$, there exists $\beta = \beta(\alpha, d) \in (0,1]$ such that the following holds: Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$ with $|\mathcal{F}| \geq d+1$. If at least $\alpha\binom{|\mathcal{F}|}{d+1}$ of the $(d+1)$-tuples in $\mathcal{F}$ are intersecting, then $\mathcal{F}$ contains an intersecting subfamily of size at least $\beta|\mathcal{F}|$.*

E-mail address: kmk90@kaist.ac.kr.

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The fractional variant of Helly's theorem first appeared as a conjecture on interval graphs, i.e. intersection graphs of families of intervals on $\mathbb{R}$. Abbott and Katchalski [1] proved that $\beta = 1 - \sqrt{1-\alpha}$ is optimal for every family whose intersection graph is a chordal graph. Note that, by a result of Gavril [10], interval graphs are chordal graphs.

The fractional Helly theorem for arbitrary dimensions was proved by Katchalski and Liu [14]. Their proof gives a lower bound $\beta \geq \alpha/(d+1)$, and also shows that β tends to 1 as α tends to 1. Note that the original theorem by Helly is obtained by setting $\alpha = 1$. Later, the quantitatively sharp value $\beta(\alpha, d) = 1 - (1-\alpha)^{1/(d+1)}$ was found by Kalai [13] and Eckhoff [7], which is a consequence of the upper bound theorem for families of convex sets.

The (p, q)-theorem, another important generalization of Helly's theorem, deals with a weaker version of the assumption, the so-called (p, q)-*condition*: for every p members in a given family, there are some q members of the family that are intersecting. For instance, the $(d+1, d+1)$-condition in $\mathbb{R}^d$ is the hypothesis of Helly's theorem. The (p, q)-theorem was proved by Alon and Kleitman [3], settling a conjecture by Hadwiger and Debrunner [11]. It states as follows.

Theorem 1.4 (*(p, q)-Theorem, Alon and Kleitman [3]*). *Let p, q and d be integers with $p \geq q \geq d+1$. Then there exists a number $HD_d(p, q)$ such that the following is true: Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$ satisfying the (p, q)-condition. Then $\mathcal{F}$ has a transversal consisting of at most $HD_d(p, q)$ points.*

The original proof of the (p, q)-theorem is quite long and involved, using various techniques. It was later shown by Alon et al. [2] that the most crucial ingredient is the fractional Helly theorem, and they showed that one can obtain a (p, q)-theorem for abstract set-systems which satisfy an appropriate "fractional Helly property". For an overview and further knowledge of this field, see the survey papers by Eckhoff [8,9] and the textbook by Matoušek [16].

Recently, Bárány et al. [6] established colorful and fractional versions of the (p, q)-theorem. A key ingredient in their proof was a colorful variant of the fractional Helly theorem.

Theorem 1.5 (*Bárány, Fodor, Montejano, Oliveros, and Pór [6]*). *Let $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{d+1}$ be finite, non-empty families (color classes) of convex sets in $\mathbb{R}^d$, and assume that $\alpha \in (0, 1]$. If at least $\alpha|\mathcal{F}_1| \cdots |\mathcal{F}_{d+1}|$ of the colorful $(d+1)$-tuples are intersecting, then some $\mathcal{F}_i$ contains an intersecting subfamily of size $\frac{\alpha}{d+1}|\mathcal{F}_i|$.*

The proof in [6] follows the standard argument where each intersecting colorful $(d+1)$-tuple is charged to one of its d-tuples. (See for instance section 8.1 in [16] for a proof of the uncolored version.)

Note that for $\alpha = 1$ we recover the hypothesis of the colorful Helly theorem, and it is natural to ask whether the function β tends to 1 as α tends to 1. This problem is implicitly contained in the paper by Bárány et al. [6] and was communicated to us by F. Fodor.

Here we solve this problem by showing the following.

Theorem 1.6. *For every $\alpha \in (0, 1]$, there exists $\beta = \beta(\alpha, d) \in (0, 1]$ tending to 1 as α tends to 1 such that the following holds: Let $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{d+1}$ be finite, non-empty families (color classes) of convex sets in $\mathbb{R}^d$. If at least $\alpha|\mathcal{F}_1| \cdots |\mathcal{F}_{d+1}|$ of the colorful $(d+1)$-tuples are intersecting, then for some $1 \leq i \leq d+1$, $\mathcal{F}_i$ contains an intersecting subfamily of size $\beta|\mathcal{F}_i|$.*

In order to prove Theorem 1.6, we will show that for every sufficiently small $\epsilon > 0$, if none of the $\mathcal{F}_i$ have an intersecting subfamily of size $(1-\epsilon)|\mathcal{F}_i|$, then there is a *positive fraction* of the colorful $(d+1)$-tuples which are *non-intersecting*. This will be done with explicit calculations.

An interesting aspect of our proof is that it is purely combinatorial (formulated in the language of uniform hypergraphs) and uses only the colorful Helly theorem as a "black box". Our method can easily be modified to provide another (simple) proof that the function β tends to 1 as α tends to 1 in the classical fractional Helly theorem (Theorem 1.3), but it does not give the optimal bound of Kalai and Eckhoff.

2. Proof of Theorem 1.6

2.1. The matching number of hypergraphs

Let $\mathcal{H}$ be an r-uniform hypergraph on a vertex set X. A subset $S \subseteq X$ is said to be an *independent set* in $\mathcal{H}$ if the induced sub-hypergraph $\mathcal{H}[S]$ contains no hyperedge. The *independence number* $\alpha(\mathcal{H})$ of $\mathcal{H}$ is the cardinality of a maximum independent set in $\mathcal{H}$. A *matching* of $\mathcal{H}$ is a set of pairwise disjoint edges in $\mathcal{H}$. The *matching number* $\nu(\mathcal{H})$ of $\mathcal{H}$ is the cardinality of a maximum matching in $\mathcal{H}$. We need the following observation.

Observation 2.1. *Let $\mathcal{H} = (X, E)$ be an r-uniform hypergraph with $|X| = n$. Suppose*

$$\alpha(\mathcal{H}) < cn$$

for some $c \in (0, 1]$. Let $M = \{e_1, \ldots, e_\nu\}$ be a maximum matching in $\mathcal{H}$. Note that $X \setminus (e_1 \cup \cdots \cup e_\nu)$ is an independent set in $\mathcal{H}$. If not, assume that there is an edge e contained in $X \setminus (e_1 \cup \cdots \cup e_\nu)$. Then $M \cup \{e\}$ is a matching in $\mathcal{H}$, which is a contradiction to the maximality of M. Thus

$$|X \setminus (e_1 \cup \cdots \cup e_\nu)| = n - r\nu(\mathcal{H}) \leq \alpha(\mathcal{H}) < cn,$$

so $\nu(\mathcal{H}) > \frac{n-cn}{r}$.