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Note

Intersection properties of maximal directed cuts in digraphs

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ABSTRACT

If D is a finite digraph, a directed cut is a subset of arcs in D having tail in some subset $X \subseteq V(D)$ and head in $V(D) \setminus X$. In this paper we prove two general results concerning intersections between maximal paths, cycles and maximal directed cuts in D. As a direct consequence of these results, we deduce that there is a path, or a cycle, in D that crosses each maximal directed cut.

1. Introduction and definitions

In this paper we always denote by $D = (V(D), A(D))$ a finite digraph (without multiple arcs and loops) having $V(D)$ as its set of vertices and $A(D)$ as its set of arcs. If a is an arc such that $a = (v, w)$, with $v, w \in V(D)$, we say that v is the *tail* of a (we denote v by $t(a)$) and that w is the *head* of a (we denote w by $h(a)$). If $B \subseteq A(D)$, we set $T(B) = \{t(a) : a \in B\}$ and $H(B) = \{h(a) : a \in B\}$. If $v \in V(D)$, we set $N^+(v) = \{z \in V(D) \setminus \{v\} : (v, z) \in A(D)\}$, $N^-(v) = \{u \in V(D) \setminus \{v\} : (u, v) \in A(D)\}$, $d^+(v) = |N^+(v)|$ and $d^-(v) = |N^-(v)|$. A *walk* in D is a sequence $W = v_0a_0v_1a_1v_2 \dots v_{n-1}a_{n-1}v_n$, where $v_0, \dots, v_n$ are vertices in D (i.e. elements of $V(D)$) and $a_0, \dots, a_{n-1}$ are arcs in D (i.e. elements of $A(D)$) such that a_i has tail in v_i and head in v_{i+1}, for $i = 0, \dots, n-1$. We say that $a_0, a_1, \dots, a_{n-1}$ are the *arcs of* W and that $v_0, v_1, \dots, v_n$ are the *vertices of* W. We set $A(W) = \{a_0, a_1, \dots, a_{n-1}\}$ and $V(W) = \{v_0, v_1, \dots, v_n\}$. When the arcs of W are clear from the context or unimportant, we will denote W simply by $v_0v_1 \dots v_n$. The *length* of W, denoted by $l(W)$, is the number of its arcs, that is n with the previous notation. A *path* is a walk whose vertices are mutually distinct. If the vertices $v_0, v_1, \dots, v_{n-1}$ are distinct, $n \geq 2$ and $v_0 = v_n$, we say that W is a *cycle*. When $W = v_0v_1 \dots v_n$ is a path or a cycle and $0 \leq i < j \leq n$, we set $W[v_i, v_j] = v_iv_{i+1} \dots v_j$ (so that, in particular, $W[v_0, v_n] = W$). Let us note that $W[v_i, v_j]$ is a path if $v_i \neq v_j$, that we call the *sub-path* of W from v_i to v_j. We say that a path P is *maximal* in D if P is not a proper sub-path of another path in D. Therefore, if $P = v_0v_1 \dots v_n$ is maximal, then $N^-(v_0) \subseteq \{v_1, \dots, v_n\}$ and $N^+(v_n) \subseteq \{v_0, \dots, v_{n-1}\}$. If X and Y are two subsets of $V(D)$, we set $(X, Y)_D = \{(x, y) \in A(D) : x \in X,\ y \in Y\}$. If $X \subseteq V(D)$, we call *directed cut* of X in D the set $(X, V(D) \setminus X)_D$, that we denote by $\xi(X)$. We say that a subset $K \subseteq A(D)$ is a *directed cut* in D if $K = \xi(X)$, for some $X \subseteq V(D)$. A directed cut K is *maximal* if there is no directed cut K' such that $K \subsetneq K'$. The directed cuts (and related arguments) for particular classes of digraphs have been studied in several works, see for example [1,2,5–9,11,12]. From an algorithmic point of view, the study of directed cuts in weighted-arc digraphs is also well studied, see for example [3,4,10].

If W is a walk in D and K is a directed cut in D, we say that W *crosses* K if $A(W) \cap K \neq \emptyset$. If $P = v_0a_0v_1a_1v_2 \dots v_{n-1}a_{n-1}v_n$ is a path and $0 \leq i < j \leq n$, we say that P has an *(i, j)-inversion* if $v_i \in N^+(v_n)$ and $v_j \in N^-(v_0)$. We say that P has an *inversion* if it has an (i, j)-inversion, for some $0 \leq i < j \leq n$. In particular, if P has a $(0, n)$-inversion then it can be extended to a cycle.

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In this paper we examine the family of the maximal directed cuts in D and we establish some links between paths, cycles and maximal directed cuts. We show how some properties of the maximal paths and cycles in D are strictly related to the family of the maximal directed cuts. A first problem that we study is the following: if we have a maximal path P in D, under what assumptions P has an inversion? There is then a non-trivial assumption, linking maximal directed cuts and maximal paths, that guarantees the existence of an inversion: if a maximal path does not cross some maximal directed cut, then the path has an inversion (see Theorem 2.2). Our second question is naturally related to the previous: in what cases a maximal path crosses each maximal directed cut? In this regard, we prove the following result: there exists a path, or a cycle, which crosses each maximal directed cut (Theorem 2.4).

2. The results

Before stating and proving the main results of this paper let us show some useful properties of maximal directed cuts.

Lemma 2.1. *Let $\xi(X)$ be a maximal directed cut in a digraph D. Then:*

(i) *For every $x \in X$, if $N^+(x) \subseteq X$, then $N^-(x) \subseteq V(D) \setminus X$.*
(ii) *For every $y \in V(D) \setminus X$, if $N^-(y) \subseteq V(D) \setminus X$, then $N^+(y) \subseteq X$.*

Proof.

(i) If $N^+(x) \subseteq X$, then $\xi(X) \subseteq \xi(X \setminus \{x\})$. By maximality of $\xi(X)$, we have $\xi(X) = \xi(X \setminus \{x\})$. It follows that $N^-(x) \subseteq V(D) \setminus X$.
(ii) If $N^-(y) \subseteq V(D) \setminus X$, then $\xi(X) \subseteq \xi(X \cup \{y\})$. By maximality of $\xi(X)$, we have $\xi(X) = \xi(X \cup \{y\})$. It follows that $N^+(y) \subseteq X$. □

In the next Theorem 2.2, we provide a sufficient condition (in terms of maximal directed cuts) for a maximal path to have an inversion.

Theorem 2.2. *Let us assume in D that a maximal path P does not cross some maximal directed cut. Then P has an inversion.*

Proof. Let $P = v_0a_0v_1a_1 \dots v_{n-1}a_{n-1}v_n$. If $v_0 \in N^+(v_n)$, then P has a $(0, n)$-inversion. Therefore, in the rest of the proof, we can assume that $v_0 \notin N^+(v_n)$. Since P is a maximal path, we have then

$$N^+(v_n) \subseteq \{v_1, \dots, v_{n-1}\} \tag{1}$$

and

$$N^-(v_0) \subseteq \{v_1, \dots, v_{n-1}\}. \tag{2}$$

Let $\xi(X)$ be a maximal directed cut in D such that

$$\xi(X) \cap A(P) = \emptyset. \tag{3}$$

We prove now that

$$n = l(P) > 1. \tag{4}$$

Let us note that, if $v_0 \in X$ and $v_1 \notin X$, then $a_0 \in \xi(X) \cap A(P)$, that contradicts (3). Thus $v_0 \notin X$ or $v_1 \in X$. Set $Y := \{v_0\} \cup X \setminus \{v_1\}$ and let $a \in \xi(X)$. If a has tail in v_0 then $a \in \xi(Y)$. Assume that v_0 is not the tail of a. Since $N^+(v_1) = N^-(v_0) = \emptyset$, a cannot have head in v_0 and a cannot have tail in v_1. Hence $a \in \xi(Y)$. On the other hand, $a_0 = (v_0, v_1) \in \xi(Y)$ but $a_0 \notin \xi(X)$ by (3). Hence $\xi(X) \subsetneqq \xi(Y)$ is verified.

Our next step is to show that

$$\{v_0, v_1, \dots, v_{n-1}\} \cap X \neq \emptyset. \tag{5}$$

If $v_0 \in X$, then obviously $\{v_0, v_1, \dots, v_{n-1}\} \cap X \neq \emptyset$. Let us suppose that $v_0 \notin X$. If $N^-(v_0) \cap X \neq \emptyset$, then, by (2), $\{v_0, v_1, \dots, v_{n-1}\} \cap X \neq \emptyset$. If $N^-(v_0) \cap X = \emptyset$, then by Lemma 2.1, part (ii), it follows that $N^+(v_0) \subseteq X$. Thus $v_1 \in N^+(v_0) \subseteq X$, so $\{v_0, v_1, \dots, v_{n-1}\} \cap X \neq \emptyset$ also in this case.

By (5), we can choose the minimum integer $k \in \{0, 1, \dots, n-1\}$ such that $v_k \in X$. It follows then that

$$\{v_k, \dots, v_n\} \subseteq X. \tag{6}$$

In fact, if (by absurd) $v_{k+1} \notin X$, then $a_k \in \xi(X) \cap A(P)$, that contradicts (3). Hence $v_{k+1} \in X$. By induction we deduce (6). In particular, by (6) we obtain

$$\{v_{n-1}, v_n\} \subseteq X. \tag{7}$$