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Note

A characterization of a class of hyperplanes of $DW(2n-1,\mathbb{F})$

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ABSTRACT

A hyperplane of the symplectic dual polar space $DW(2n-1,\mathbb{F})$, $n \geq 2$, is said to be of subspace-type if it consists of all maximal singular subspaces of $W(2n-1,\mathbb{F})$ meeting a given $(n-1)$-dimensional subspace of $\mathrm{PG}(2n-1,\mathbb{F})$. We show that a hyperplane of $DW(2n-1,\mathbb{F})$ is of subspace-type if and only if every hex F of $DW(2n-1,\mathbb{F})$ intersects it in either F, a singular hyperplane of F or the extension of a full subgrid of a quad. In the case $\mathbb{F}$ is a perfect field of characteristic 2, a stronger result can be proved, namely a hyperplane H of $DW(2n-1,\mathbb{F})$ is of subspace-type or arises from the spin-embedding of $DW(2n-1,\mathbb{F}) \cong DQ(2n,\mathbb{F})$ if and only if every hex F intersects it in either F, a singular hyperplane of F, a hexagonal hyperplane of F or the extension of a full subgrid of a quad.

1. Introduction

Hyperplanes have been investigated for several classes of point-line geometries. In particular, many constructions and classification results have been obtained, and the close relationship between hyperplanes and projective embeddings has been studied. This relationship originates from the fact that with every full embedding e of a point-line geometry $\mathcal{S}$ there are associated hyperplanes, the so-called hyperplanes of $\mathcal{S}$ arising from e. The connection between hyperplanes and projective embeddings has played a crucial role in Tits' classification of polar spaces [21]. The question which hyperplanes of a given point-line geometry arise from a full projective embedding has been widely investigated (see e.g. Cohen and Shult [4, Theorem 5.12] for the case of polar spaces). Sometimes hyperplanes tell you whether a point-line geometry can admit a full projective embedding (see e.g. Ronan [16, Corollary 2, p. 183]) or whether a given full projective embedding is absolutely universal (see e.g. Shult [18]).

This note is concerned with characterizing certain hyperplanes of dual polar spaces. Hyperplanes of dual polar spaces are usually characterized in terms of their possible intersections with convex subspaces. The initial characterization results used the possible intersections with quads[1] as basis for the characterizations. In this regard, it is worth mentioning the work of Shult & Thas [20], Pasini & Shpectorov [12], Cooperstein & Pasini [5], Cardinali, De Bruyn & Pasini [3] and De Bruyn [7] on locally singular, locally subquadrangular and locally ovoidal hyperplanes. Pralle [14] investigated hyperplanes in dual polar spaces of rank 3 that do not admit subquadrangular quads and those without singular quads (for arbitrary ranks) were studied in [13]. In joint work with the author [10], he also investigated hyperplanes of symplectic dual polar spaces of rank 3 without ovoidal quads. This classification was later extended by the author to arbitrary ranks [6].

There are also a number of characterizations in terms of the possible intersections with hexes. In this regard, it is worth mentioning the result of Cardinali, De Bruyn and Pasini [3, Lemma 3.4] who showed that the singular hyperplanes of thick dual polar spaces are precisely the hyperplanes intersecting each hex F in either F or a singular hyperplane of $\widetilde{F}$. Pralle and Shpectorov [15] studied hyperplanes in thick dual polar spaces of rank 3 intersecting each hex in the extension of an ovoid

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1 Many notions used in this introductory section will be explicitly defined in Section 2.

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of a quad. In [8], the author extended this study to hyperplanes in dual polar spaces of arbitrary rank that intersect each hex F in either F, a singular hyperplane of $\widetilde{F}$ or the extension of an ovoid of a quad. It was shown there that these hyperplanes are precisely the possible trivial extensions of the SDPS-hyperplanes, a class of hyperplanes introduced in [11].

In this note, we consider a problem that is similar to the one studied in [8]. We take a look at hyperplanes of dual polar spaces that intersect each hex F in either F, a singular hyperplane of $\widetilde{F}$ or the extension of a subquadrangle of a quad. As we will see, it is possible to classify all these hyperplanes in the case of symplectic dual polar spaces. They are precisely the hyperplanes of subspace-type, a class of hyperplanes under investigation in [9]. In the case where the field $\mathbb{F}$ is perfect of characteristic 2, it is even possible to classify all hyperplanes if one allows an additional possibility for the intersection with hexes. The following two results are the main results of this note.

Theorem 1.1. *The following are equivalent for a hyperplane H of $DW(2n-1, \mathbb{F})$, $n \geq 3$:*

(1) *H is a hyperplane of subspace-type;*
(2) *for every hex F of $DW(2n-1, \mathbb{F})$, $F \cap H$ is either F, a singular hyperplane of $\widetilde{F}$ or the extension of a full subgrid of a quad of $\widetilde{F}$.*

Theorem 1.2. *Let $n \geq 3$ and $\mathbb{F}$ a perfect field of characteristic 2. Then the following are equivalent for a hyperplane H of $DW(2n-1, \mathbb{F})$:*

(1) *H is either a hyperplane of subspace-type or arises from the spin-embedding of $DW(2n-1, \mathbb{F}) \cong DQ(2n, \mathbb{F})$;*
(2) *for every hex F of $DW(2n-1, \mathbb{F})$, $F \cap H$ is either F, a singular hyperplane of $\widetilde{F}$, a hexagonal hyperplane of $\widetilde{F}$ or the extension of a full subgrid of a quad of $\widetilde{F}$.*

It is somewhat unfortunate that the proofs of Theorems 1.1 and 1.2 are for a large extent already contained in [6]. This note could therefore also be seen as an addendum to [6]. The main purpose of [6] was to extend the classification result of [10] to arbitrary ranks. By focussing on this particular goal, we overlooked[2] then that the proof can be modified to a proof of our main results. We fear that this fact might remain unnoticed by a future reader, as this modification still requires some work. Indeed, certain arguments in [6] are not relevant for the current treatment and other arguments do not work when the underlying field is infinite (due to the use of counting arguments) or of order 2. These problems will be by-passed here by means of alternative arguments and a change of the order of the intermediate lemmas that will moreover lead to a simplification. For convenience of the reader, we still mention the whole chain of lemmas leading to the proofs of Theorems 1.1 and 1.2. The proofs of those lemmas that are basically contained in [6] will be omitted and instead an explicit reference to [6] will be given.

2. Preliminaries

With every polar space Π of rank $n \geq 2$ (in the sense of Tits [21, Chapter 7]) there is associated a dual polar space Δ of rank n. This is the point-line geometry whose points and lines are the maximal and next-to-maximal singular subspaces of Π, with incidence being reverse containment. Distances $\mathrm{d}(\cdot, \cdot)$ between points of Δ will always be measured in the collinearity graph which has diameter n.

There exists a bijective correspondence between the nonempty convex subspaces of Δ and the singular subspaces of Π: if α is a singular subspace of Π, then the set F_α consisting of all maximal singular subspaces containing α is a convex subspace of Δ. The convex subspaces of diameter 2, 3 and $n-1$ are called the *quads*, *hexes* and *maxes*, respectively. If F is a convex subspace of diameter $\delta \geq 2$ of Δ, then the point-line geometry $\widetilde{F}$ induced on F is a dual polar space of rank δ. In particular, if Q is a quad, then $\widetilde{Q}$ is a dual polar space of rank 2, i.e. a generalized quadrangle. Two points x and y of Δ at distance δ from each other are contained in a unique convex subspace $\langle x, y\rangle$ of diameter δ.

If F is a convex subspace of Δ and x is a point, then there exists a (necessarily unique) point $\pi_F(x) \in F$ such that $\mathrm{d}(x, y) = \mathrm{d}(x, \pi_F(x)) + \mathrm{d}(\pi_F(x), y)$ for every point $y \in F$. In particular, for every point-line pair (x, L), there is a unique point on L nearest to x. The convex subspaces through a given point x, ordered by inclusion, define a projective space $Res(x)$ of dimension $n-1$.

Suppose H is a hyperplane of Δ, i.e. a proper subspace meeting each line. If x is a point of H, then $\Lambda_H(x)$ denotes the set of lines through x contained in H. We will often regard $\Lambda_H(x)$ as a set of points of $Res(x)$. If F is a convex subspace of Δ, then either $F \subseteq H$ (in which case F is called *deep* with respect to H) or $F \cap H$ is a hyperplane of $\widetilde{F}$. As there are three types of hyperplanes in generalized quadrangles (ovoids, subquadrangles and perps of points), we see that for every quad Q of Δ, one of the following cases occurs:

(1) Q is deep, i.e. contained in H;
(2) $Q \cap H = x^\perp \cap Q$ for a certain point $x \in Q$;
(3) $Q \cap H$ is a full subquadrangle of $\widetilde{Q}$;
(4) $Q \cap H$ is an ovoid of $\widetilde{Q}$, i.e. a set of points meeting each line of $\widetilde{Q}$ in a singleton.

[2] The classification result obtained in [6], namely those of hyperplanes of $DW(2n-1, \mathbb{F})$ without ovoidal quads, is only valid when $\mathbb{F} = \mathbb{F}_q$ for a certain prime power $q \neq 2$. For infinite fields or the smallest field $\mathbb{F}_2$, other examples of such hyperplanes exist and a complete classification was and is still missing. As such, there was no need in [6] to let intermediate results work for any field.