Discrete Mathematics 340 (2017) 3183–3196

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Geodesics on the regular tetrahedron and the cube

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ARTICLE INFO

Article history:
Received 20 August 2015
Received in revised form 5 July 2016
Accepted 12 July 2016
Available online 10 August 2016

Keywords:
Geodesic
Cube
Regular tetrahedron
Stern–Brocot tree

ABSTRACT

Consider all geodesics between two given points on a polyhedron. On the regular tetrahedron, we describe all the geodesics from a vertex to a point, which could be another vertex. Using the Stern–Brocot tree to explore the recursive structure of geodesics between vertices on a cube, we prove, in some precise sense, that there are twice as many geodesics between certain pairs of vertices than other pairs. We also obtain the fact that there are no geodesics that start and end at the same vertex on the regular tetrahedron or the cube.

1. Introduction

A **geodesic curve** γ on a surface embedded in $\mathbf{R}^3$ is a curve such that for any two sufficiently close points p and q of γ, the shortest path on the surface between p and q coincides with γ. We consider the following problem: Given a point p on a polyhedron and a vertex v, what are the geodesics from p to v? In this paper, we focus on the regular tetrahedron and the cube, giving special attention to the class of vertex-to-vertex geodesics, that is, where p is also a vertex. Even though geodesics on convex polyhedra do not pass through vertices, it is possible for a geodesic arc to start and end at the same vertex (a "geodesic loop"). However, it turns out that this does not happen for the regular tetrahedron or for the cube (see Corollaries 3.8 and 5.15, respectively), so we consider p distinct from v.

1.1. Previous results

Fuchs and Fuchs [4] studied closed geodesics on regular polyhedra. They give the (previously known, simple) result for the regular tetrahedron and also describe all closed geodesics on the cube and regular octahedron. For the regular tetrahedron [4, §3], they show that every geodesic is non-self-intersecting and give a simple characterization of every closed geodesic. Since it does not intersect itself, every closed geodesic cuts the surface of the tetrahedron into two pieces. For the cube [4, §4], they show that, up to symmetry and translation, there are exactly three non-self-intersecting closed geodesics, two of which are planar. They also show that a self-intersecting closed geodesic intersects itself perpendicularly. They exhibit all three closed non-self-intersecting geodesics and show several examples of longer geodesics with many self intersections. Recently, D. Fuchs connected these results on closed geodesics of regular polyhedra to the problem considered in our present paper, and gave different proofs of some of our results [3].

More generally, over the past century many authors have studied straight paths on polyhedra. Alexandrov showed in 1950 that a shortest path never passes through a vertex of positive curvature [1]. Similarly, Sharir and Schorr showed that on a

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http://dx.doi.org/10.1016/j.disc.2016.07.004
0012-365X/

convex polyhedron, shortest paths do not pass through vertices [10], although they may do so on a non-convex polyhedron [7]. Moreover, shortest paths never self-intersect and never intersect the same face more than once. Locally, closed geodesics are shortest paths and therefore inherit some of these properties. In particular, a closed geodesic cannot pass through a vertex. In contrast, a geodesic loop can start and end at the same vertex, but its ends cannot join smoothly to form a locally shortest path at the vertex. These results are discussed in Demaine and O'Rourke's comprehensive book [2, §24].

Nikonorov and Nikonorova measured lengths of shortest paths on parallelepipeds, using techniques similar to ours [8, Figure 3]. Itoh, O'Rourke, and Vîlcu cut up and unfold polyhedra, using geodesics and quasigeodesics (a notion slightly weaker than that of geodesics) to define where the polyhedral boundary is cut, in order to unfold the polyhedral boundary into a single flat polygon [5]. They work to ensure that the polyhedral boundary is in 1-to-1 correspondence with the flattened polygon, whereas our method of unwrapping the cube in Section 5 is certainly not 1-to-1. While our study of tetrahedra in this paper is restricted to regular tetrahedra, Rouyer and Vîlcu study the geometry of non-regular tetrahedra in a very general and abstract way [9].

1.2. Results presented in this paper

In Section 3, we characterize all of the geodesics from any point to any vertex on the regular tetrahedron.

Theorem 3.6: We describe the directions from a given point p in which a geodesic will end at a given vertex v.

Corollary 3.7: Given a pair of (necessarily distinct) vertices v_0 and v, we give a complete description of the directions of geodesics from v_0 to v.

In Section 4, we introduce our conventions and give basic results about geodesics on the cube.

In Section 5, we develop the beautiful recursive structure underlying vertex-to-vertex paths on the cube, based on the Stern–Brocot tree. We use this structure to derive several results:

Theorem 5.17: There are twice as many geodesics to the cube vertex at greatest distance from the starting vertex as there are to each of the three vertices that are diagonally opposite the starting vertex along a common face, and 1.5 times as many as to the three adjacent vertices. (The notion of "twice as many" is made rigorous below.)

Corollary 5.18: We count the number of geodesics to a given vertex, depending on the "depth" in the Stern–Brocot tree.

In Section 6, we consider geodesics starting from a general point in a face by associating them with line segments from a point in the unit square to a lattice point in $\mathbf{R}^2$.

Lemma 6.1: We characterize the lattice points that are reachable ("visible") from a starting point p in a face.

Proposition 6.3: We give an algorithm for determining the "tumble sequence" to a given lattice point.

2. Basic definitions

Given a polyhedron, a **geodesic** is a locally shortest curve on its surface.

Our goal is: Given a polyhedron S, a distinguished vertex v, and a distinguished point p (which may or may not be a vertex), determine all of the geodesics from p to v. We consider a ray starting at p, following the surface of S, possibly wrapping around S many times, before finally arriving at v. To do this, we **unfold** the faces of S in the following way. For concreteness, suppose that the face containing p is on the xy-plane. When the ray hits an edge e, we **tumble** S by applying the unique orientation-preserving isometry that fixes edge e and places the adjacent face of S on the xy-plane on the other side of e.[1] After the tumble, the geodesic continues on this new face. These two segments in the xy-plane form a straight line segment by definition. We continue in like manner until the geodesic hits a vertex. In such a way, a geodesic on S naturally corresponds to a straight line segment in the xy-plane, contained in a strip formed by copying the faces of S to the xy-plane as S tumbles. Note that this unfolding depends on the geodesic being considered.

In this paper, we study special cases of this polyhedron problem: the regular tetrahedron and the cube. These polyhedra are especially elegant because their faces tile the plane. When either of these polyhedra is tumbled in all possible ways, the points in the xy-plane touched by the vertices form a lattice Λ: the equilateral triangle lattice for the tetrahedron and the square lattice for the cube.

Definition 2.1. Given a point $p \in \mathbf{R}^2$, a lattice point $q \in \Lambda$ distinct from p is **visible** (**from** p) if the interior of the segment $\overline{pq}$ from p to q does not contain any lattice points of Λ (see Fig. 5.1).

Lemma 2.2. *Let S be the regular tetrahedron or the cube and Λ the corresponding lattice. A segment $\overline{pq}$ on the xy-plane corresponds to a geodesic of S ending at a vertex if and only if $q \in \Lambda$ is a visible lattice point.*

Proof. We reconstruct the corresponding geodesic given the segment on the plane: each time the segment leaves the current face through an edge, we tumble S across that edge. The added vertex of this new face is of course a lattice point. If q is not a lattice point, then the corresponding geodesic does not end at a vertex. On the other hand, if q is a lattice point that is not visible, then the corresponding geodesic passes through a vertex, a contradiction. □

We identify the geodesics on S with the corresponding segments in the xy-plane (see Fig. 4.2).

[1] Think of this as rolling a polyhedral die on a tabletop without slipping.