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# Spectra and Laplacian spectra of arbitrary powers of lexicographic products of graphs

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ABSTRACT

Consider two graphs $G$ and $H$. Let $H^k[G]$ be the lexicographic product of $H^k$ and $G$, where $H^k$ is the lexicographic product of the graph $H$ by itself $k$ times. In this paper, we determine the spectrum of $H^k[G]$ and $H^k$ when $G$ and $H$ are regular and the Laplacian spectrum of $H^k[G]$ and $H^k$ for $G$ and $H$ arbitrary. Particular emphasis is given to the least eigenvalue of the adjacency matrix in the case of lexicographic powers of regular graphs, and to the algebraic connectivity and the largest Laplacian eigenvalues in the case of lexicographic powers of arbitrary graphs. This approach allows the determination of the spectrum (in case of regular graphs) and Laplacian spectrum (for arbitrary graphs) of huge graphs. As an example, the spectrum of the lexicographic power of the Petersen graph with the googol number (that is, $10^{100}$ ) of vertices is determined. The paper finishes with the extension of some well known spectral and combinatorial invariant properties of graphs to its lexicographic powers.



## 1. Introduction

The lexicographic product of a graph $H$ with itself several times is a very special graph product, it is a kind of fractal graph which reproduces its copy in each of the positions of its vertices and connects all the vertices of each copy with another copy when they are placed in positions corresponding to adjacent vertices of $H$. This procedure can be repeated, reproducing a copy of the previous iterated graph in each of the positions of the vertices of $H$ and so on. Despite the spectrum and Laplacian spectrum of the lexicographic product of two graphs (with some restrictions regarding the spectrum) expressed in terms of the two factors are well known (see [2], where a unified approach is given), it is not the case of the spectra and Laplacian spectra of graphs obtained by iterated lexicographic products, herein called lexicographic powers, of regular and arbitrary graphs, respectively. A lexicographic power $H^k$ of a graph $H$ can produce a graph with a huge number of vertices whose spectra and Laplacian spectra may not be determined using their adjacency and Laplacian matrices, respectively. The expressions herein deduced for the spectra and Laplacian spectra of lexicographic powers can be easily programmed, for example, in Mathematica, and the results can be obtained immediately. For instance, the spectrum of the 100th lexicographic power of the Petersen graph, presented in Section 3, was obtained by Mathematica and the computations lasted only a few seconds. Notice that such lexicographic power has the googol number (that is, $10^{100}$) of vertices.

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The paper is organized as follows. In the next section, the notation is introduced and some preliminary results are given. The main results are introduced in Section 3, where the spectra (Laplacian spectra) of $H^k[G]$ and $H^k$, when $G$ and $H$ are regular (arbitrary) graphs, are deduced. Particular attention is given to the Laplacian index and algebraic connectivity of the lexicographic powers of arbitrary graphs. In Section 4, the obtained results are applied to extend some well known properties and spectral relations of combinatorial invariants of graphs $H$ to its lexicographic powers $H^k$.

## 2. Preliminaries

In this work we deal with simple and undirected graphs. If $G$ is such a graph of order $n$, its vertex set is denoted by $V(G)$ and its edge set by $E(G)$. The elements of $E(G)$ are denoted by $ij$, where $i$ and $j$ are the extreme vertices of the edge $ij$. The degree of $j \in V(G)$ is denoted by $d_G(j)$, the minimum and maximum degree of the vertices in $G$ are $\delta(G)$ and $\Delta(G)$ and the set of the neighbors of a vertex $j$ is $N_G(j)$. The *adjacency matrix* of $G$ is the $n \times n$ matrix $A_G$ whose $(i,j)$-entry is equal to 1 whether $ij \in E(G)$ and 0 otherwise. The *Laplacian matrix* of $G$ is the matrix $L_G = D - A_G$, where $D$ is the diagonal matrix whose diagonal elements are the degrees of the vertices of $G$. Since $A_G$ and $L_G$ are symmetric matrices, their eigenvalues are real numbers. From Geršgorin's theorem, the eigenvalues of $L_G$ are nonnegative. The multiset (that is, the set with possible repetitions) of eigenvalues of a matrix $M$ is called the *spectrum* of $M$ and denoted $\sigma(M)$. Throughout the paper, we write $\sigma_A(G) = \{\lambda_1^{[g_1]}, \dots, \lambda_s^{[g_s]}\}$ (respectively, $\sigma_L(G) = \{\mu_1^{[l_1]}, \dots, \mu_t^{[l_t]}\}$) when $\lambda_1 > \cdots > \lambda_s$ ($\mu_1 > \cdots > \mu_t$) are the distinct eigenvalues of $A_G$ ($L_G$) indexed in decreasing order — in this case, $\gamma^{[r]}$ means that the eigenvalue $\gamma$ has multiplicity $r$. If convenient, we write $\gamma(G)$ in place of $\gamma$ to indicate an eigenvalue of a matrix associated to $G$, and we denote the eigenvalues of $A_G$ (respectively, $L_G$) indexed in non increasing order, as $\lambda_1(G) \geq \cdots \geq \lambda_n(G)$ ($\mu_1(G) \geq \cdots \geq \mu_n(G)$).

As usual, the adjacency matrix eigenvalues of a graph $G$ are called the *eigenvalues of* $G$. We remember that $\mu_n(G) = 0$ (the all one vector is the associated eigenvector) and its multiplicity is equal to the number of components of $G$. Besides, $\mu_{n-1}(G)$ is called the *algebraic connectivity* of $G$ [7]. Further concepts not defined in this paper can be found in [3,5].

The *lexicographic product* (also called the *composition*) of the graphs $H$ and $G$ is the graph $H[G]$ (also denoted by $H \circ G$) for which the vertex set is the cartesian product $V(H) \times V(G)$ and such that a vertex $(x_1, y_1)$ is adjacent to the vertex $(x_2, y_2)$ whenever $x_1$ is adjacent to $x_2$ or $x_1 = x_2$ and $y_1$ is adjacent to $y_2$ (see [13] and [15] for notations and further details). This graph operation was introduced by Harary in [11] and Sabidussi in [18]. It is immediate that the lexicographic product is associative but not commutative.

The lexicographic product was generalized in [19] under the designation of *generalized composition* as follows: consider a graph $H$ of order $n$ and graphs $G_i$, $i = 1, \dots, n$, with vertex sets $V(G_i)$s two by two disjoints is the graph such that

$$V(H[G_1, \dots, G_n]) = \bigcup_{i=1}^{n} V(G_i) \quad \text{and}$$

$$E(H[G_1, \dots, G_n]) = \bigcup_{i=1}^{n} E(G_i) \cup \bigcup_{ij \in E(H)} E(G_i \vee G_j),$$

where $G_i \vee G_j$ denotes the join of the graphs $G_i$ and $G_j$. This operation is called in [2] the $H$*-join* of graphs $G_1, \dots, G_n$. In [19] and [2], the spectrum of $H[G_1, \dots, G_n]$ is provided, where $H$ is an arbitrary graph and $G_1, \dots, G_n$ are regular graphs. Furthermore, in [8] and [2], using different approaches, the spectrum of the Laplacian matrix of $H[G_1, \dots, G_n]$ for arbitrary graphs was characterized. The Laplacian spectrum of the $H$-join was previously obtained in [17] in the particular case of a graphs with tree structure (that is, when $H$ is a tree).

Let $H$ be a graph of order $n$ and $G$ be an arbitrary graph. If, for $1 \leq i \leq n$, $G_i$ is isomorphic to $G$, it follows immediately that $H[G_1, \dots, G_n] = H[G]$, a fact also noted in [1].

Now, let us focus on the spectrum of the adjacency and Laplacian matrix of the above generalized graph composition.

Assuming that $G_1, \dots, G_n$ are all isomorphic to a particular graph $G$ (which should be regular in the case of Corollary 2.1), as consequence of Theorems 5 and 8 in [2], we have the following corollaries.

**Corollary 2.1.** *Let $H$ be a graph of order $n$ with $\sigma_A(H) = \{\lambda_1^{[h_1]}(H), \dots, \lambda_t^{[h_t]}(H)\}$, where the superscript $[h_i]$ stands for the multiplicity of the eigenvalue $\lambda_i(H)$, and let $G$ be a $p$-regular graph of order $m$ with $\sigma_A(G) = \{\lambda_1^{[g_1]}(G), \dots, \lambda_s^{[g_s]}(G)\}$. Then*

$$\sigma_A(H[G]) = \{p^{[n(g_1-1)]}, \dots, \lambda_s^{[ng_s]}(G)\} \cup \{(m\lambda_1(H) + p)^{[h_1]}, \dots, (m\lambda_t(H) + p)^{[h_t]}\}.$$

**Corollary 2.2.** *Let $H$ be a graph of order $n$ with $\sigma_L(H) = \{\mu_1(H), \dots, \mu_n(H)\}$ and let $G$ be a graph of order $m$ with $\sigma_L(G) = \{\mu_1(G), \dots, \mu_m(G)\}$. Then*

$$\sigma_L(H[G]) = \left(\bigcup_{j=1}^{n} \{md_H(j) + \mu_i(G) : 1 \leq i \leq m-1\}\right) \cup \{m\mu_1(H), \dots, m\mu_n(H)\}.$$