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Note

A revised Moore bound for mixed graphs

Dominique Buset [a], Mourad El Amiri [a], Grahame Erskine [b,*], Mirka Miller [c], Hebert Pérez-Rosés [d]

[a] Université Libre de Bruxelles, Belgium
[b] Open University, Milton Keynes, UK
[c] University of Newcastle, NSW, Australia
[d] University of Lleida, Spain

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ABSTRACT

The degree-diameter problem seeks to find the maximum possible order of a graph with a given (maximum) degree and diameter. It is known that graphs attaining the maximum possible value (the *Moore bound*) are extremely rare, but much activity is focused on finding new examples of graphs or families of graph with orders approaching the bound as closely as possible.

There has been recent interest in this problem as it applies to *mixed* graphs, in which we allow some of the edges to be undirected and some directed. A 2008 paper of Nguyen and Miller derived an upper bound on the possible number of vertices of such graphs. We show that for diameters larger than three, this bound can be reduced and we present a corrected Moore bound for mixed graphs, valid for all diameters and for all combinations of undirected and directed degrees.

1. Introduction

The Degree Diameter Problem for graphs has its motivation in the efficient design of interconnection networks. We seek to find the largest possible order (number of vertices) of a graph given constraints on its diameter k and the maximum degree d of any of its vertices. Often the problem is studied for undirected graphs, in which case the well-known *Moore bound* (see e.g. [4]) states that the number of vertices cannot exceed

$$M_{d,k} = \begin{cases} 1 + d\dfrac{(d-1)^k - 1}{d-2} & \text{if } d > 2 \\ 2k+1 & \text{if } d = 2. \end{cases} \tag{1}$$

In the undirected case it is well known [4] that no graph can achieve this bound if $k > 2$, and for diameter 2 the only known *Moore graphs* correspond to degrees 2, 3 and 7, with the case $d = 57$ being a famous open problem.

In the case of directed graphs the corresponding Moore bound [4] is given by

$$M_{d,k} = \begin{cases} \dfrac{d^{k+1} - 1}{d-1} & \text{if } d > 1 \\ k+1 & \text{if } d = 1. \end{cases} \tag{2}$$

* Corresponding author.
E-mail address: grahame.erskine@open.ac.uk (G. Erskine).

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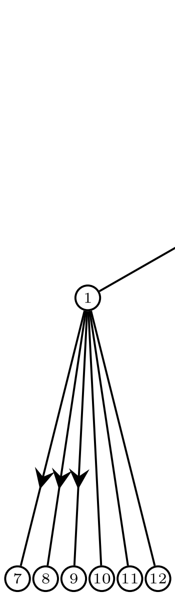

Fig. 1. The labelled Moore tree for $z = 3, r = 3, k = 2$.

In this paper we are concerned with the *mixed* or *partially directed* problem, in which we allow some of the edges in our graph to be directed and some undirected. We conform to the most usual notation in the literature, so that the maximum undirected degree of a vertex (the number of undirected edges incident to it) is denoted by r. The maximum directed degree is taken to mean the maximum number of out-arcs from any vertex and is denoted by z. As usual we denote the diameter of a graph by k.

To bound the maximum possible number of vertices, the approach is to consider a spanning tree rooted at some arbitrary vertex. It is not difficult to see that maximality is only achieved when each vertex has a unique parent at the previous level in the tree, and the maximum possible number of neighbours at the next level. Fig. 1 shows such a tree for the case $z = 3, r = 3, k = 2$.

Note that this Moore bound is only attained in a very small number of known cases. Nguyen, Miller and Gimbert [6] show that no graphs attaining the bound exist if the diameter $k \geq 3$. For $k = 2$, the known examples [4] are a family of Kautz graphs with $r = 1, z \geq 1$ and a graph of Bosák with $r = 3, z = 1$. Recently, Jørgensen [1] has discovered a pair of graphs with $r = 3, z = 7$.

In [5] the general upper bound for the order of a graph with parameters z, r, k is given as

$$M_{z,r,k} = 1 + (z + r) + z(z + r) + r(z + r - 1) + \cdots + z(z + r)^{k-1} + r(z + r - 1)^{k-1}. \quad (3)$$

It seems that this formula may have been extrapolated from the expressions for graphs of small diameter. In this paper we develop a precise formula for the Moore bound, and show that for all diameters greater than 3 this is in fact smaller than the bound stated in [5].

2. Revised Moore bound

Theorem 1. *Let $M_{z,r,k}$ denote the largest possible number of vertices in a mixed graph of diameter k, maximum directed degree z and maximum undirected degree r. Then:*

$$M_{z,r,k} = A\frac{u_1^{k+1} - 1}{u_1 - 1} + B\frac{u_2^{k+1} - 1}{u_2 - 1} \quad (4)$$

where:

$$v = (z + r)^2 + 2(z - r) + 1$$
$$u_1 = \frac{z + r - 1 - \sqrt{v}}{2}$$
$$u_2 = \frac{z + r - 1 + \sqrt{v}}{2}$$