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Hamiltonian paths in k-quasi-transitive digraphs

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ABSTRACT

Let $D = (V(D), A(D))$ be a digraph and k be an integer with $k \geq 2$. A digraph D is k-quasi-transitive, if for any path $x_0x_1 \ldots x_k$ of length k, x_0 and x_k are adjacent. In this paper, we consider the traceability of k-quasi-transitive digraphs with even $k \geq 4$. We prove that a strong k-quasi-transitive digraph D with even $k \geq 4$ and $\operatorname{diam}(D) \geq k+2$ has a Hamiltonian path. Moreover, we show that a strong k-quasi-transitive digraph D such that either k is odd or $k = 2$ or $\operatorname{diam}(D) < k + 2$ may not contain Hamiltonian paths.

1. Terminology and introduction

We shall assume that the reader is familiar with the standard terminology on digraphs and refer the reader to [1] for terminology not defined here. We only consider finite digraphs without loops or multiple arcs. Let D be a digraph with vertex set $V(D)$ and arc set $A(D)$. For any $x, y \in V(D)$, we will write $x \to y$ if $xy \in A(D)$, and also, we will write $\overline{xy}$ if $x \to y$ or $y \to x$. For disjoint subsets X and Y of $V(D)$ or subdigraphs of D, $X \to Y$ means that every vertex of X dominates every vertex of Y, $X \Rightarrow Y$ means that there is no arc from Y to X and $X \mapsto Y$ means that both of $X \to Y$ and $X \Rightarrow Y$ hold. For subsets X, Y of $V(D)$, we define $(X, Y) = \{xy \in A(D) : x \in X, y \in Y\}$. If $X = \{x\}$, then we write (x, Y) instead of $(\{x\}, Y)$. Likewise, if $Y = \{y\}$, then we write (X, y) instead of $(X, \{y\})$. Let D' be a subdigraph of D and $x \in V(D) \setminus V(D')$. We say that x and D' are adjacent if x and some vertex of D' are adjacent. For $S \subseteq V(D)$, we denote by $D[S]$ the subdigraph of D induced by the vertex set S. The converse of D is the digraph which one obtains from D by reversing all arcs.

Let x and y be two vertices of $V(D)$. The *distance* from x to y in D, denoted $d(x, y)$, is the minimum length of an (x, y)-path, if y is reachable from x, and otherwise $d(x, y) = \infty$. The distance from a set X to a set Y of vertices in D is $d(X, Y) = \max\{d(x, y) : x \in X, y \in Y\}$. The *diameter* of D is $\operatorname{diam}(D) = d(V(D), V(D))$. Clearly, D has finite diameter if and only if it is strong.

Let $P = y_0y_1 \ldots y_k$ be a path or a cycle of D. For $i < j$, $y_i, y_j \in V(P)$ we denote by $P[y_i, y_j]$ the *subpath* of P from y_i to y_j. Let $Q = q_0q_1 \ldots q_n$ be a vertex-disjoint path or cycle with P in D. If there exist $y_i \in V(P)$ and $q_j \in V(Q)$ such that $y_iq_j \in A(D)$, then we will use $P[y_0, y_i]Q[q_j, q_n]$ to denote the path $y_0y_1 \ldots y_iq_jq_{j+1} \ldots q_n$. Let C be a cycle of length k and $V_1, V_2, \ldots, V_k$ be pairwise disjoint vertex sets. *The extended cycle* $C[V_1, V_2, \ldots, V_k]$ is the digraph with vertex set $V_1 \cup V_2 \cup \cdots \cup V_k$ and arc set $\cup_{i=1}^{k}\{v_iv_{i+1} : v_i \in V_i, v_{i+1} \in V_{i+1}\}$, where subscripts are taken modulo k. That is, we have $V_1 \mapsto V_2 \mapsto \ldots \mapsto V_k \mapsto V_1$ and there are no other arcs in this extended cycle.

A digraph is quasi-transitive, if for any path $x_0x_1x_2$ of length 2, x_0 and x_2 are adjacent. The concept of k-quasi-transitive digraphs was introduced in [5] as a generalization of quasi-transitive digraphs. A digraph is *k-quasi-transitive*, if for any path $x_0x_1 \ldots x_k$ of length k, x_0 and x_k are adjacent. The k-quasi-transitive digraphs have been studied in [5,3,7,6].

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A digraph D is traceable if D possesses a Hamiltonian path. A digraph D is unilateral if, for every pair x, y of vertices of D, x is reachable from y or y is reachable from x (or both). A path P is unilateral; being unilateral is a necessary condition for traceability of digraphs. Clearly, every strong digraph is unilateral. In this paper, we consider the traceability of strong k-quasi-transitive digraphs. By the definition of k-quasi-transitive digraphs, a semicomplete bipartite digraph must be a k-quasi-transitive digraph with odd k. Clearly a semicomplete bipartite digraph $D = (V_1, V_2)$ with $|V_1| - |V_2| \geq 2$ has no Hamiltonian path. Hence we only consider the traceability of strong k-quasi-transitive digraphs with even k.

It can be shown that a strong k-quasi-transitive digraph with $\mathrm{diam}(D) \leq k + 1$ may not contain Hamiltonian paths. For example, see the following three digraphs. Let the digraph $D_1 = C_{k+1}[V_1, V_2, \ldots, V_{k+1}]$ with $V_1 = \{x_1, x_1', x_1''\}$ and $V_i = \{x_i\}$ for $i \in \{2, 3, \ldots, k+1\}$. Observe that $d_{D_1}(x_1, x_1') = k+1$ and $d_{D_1}(x, y) \leq k+1$ for any $x, y \in V(D_1)$. Hence $\mathrm{diam}(D_1) = k+1$. Let the digraph $D_2 = D_1 \cup \{x_3x_1, x_3x_1', x_3x_1''\}$. Observe that $d_{D_2}(x_{k+1}, x_k) = k$ and $d_{D_2}(x, y) \leq k$ for any $x, y \in V(D_2)$. Hence $\mathrm{diam}(D_2) = k$. Let the digraph $D_3 = C_s[V_1, V_2, \ldots, V_s]$ with $|V_1| \geq 3$, $|V_i| = 1$ for $i \in \{2, 3, \ldots, s\}$ and $s \leq k - 1$. Note that $\mathrm{diam}(D_3) = s \leq k - 1$. It is not difficult to see that the digraphs D_1, D_2 and D_3 are all strong k-quasi-transitive digraphs and do not poses any Hamiltonian path.

It can also be shown that a strong quasi-transitive digraph with $\mathrm{diam}(D) = 4$ may not contain Hamiltonian paths. For example, see the following digraph. Denote a digraph D_4 with vertex set $\{x_0, x_1, x_2, x_3, x, y, z\}$ and arc set $\{x_0x_1, x_1x_2, x_2x_3, x_3x_0, x_3x_1, x_2x_0\} \cup \{xx_i, yx_i, zx_i, x_3x, x_3y, x_3z\}$ for $i \in \{0, 1, 2\}$. It is easy to check that D_4 is a quasi-transitive digraph. Observe that $d_{D_4}(x_0, x) = 4$ and $d_{D_4}(x, y) \leq 4$ for any $x, y \in V(D_4)$. Hence $\mathrm{diam}(D_4) = 4$. If P is a Hamiltonian path in D_4, then one of x, y and z must be an intermediate vertex of P, say x. Hence $x_3x \in A(P)$ and so $x_3y, x_3z \notin A(P)$. Combining this with $d^-(y) = d^-(z) = 1$, we have y and z are both the initial vertex of P, a contradiction. Thus D_4 has not Hamiltonian paths. In Section 2, we shall show that a strong k-quasi-transitive digraph D with even $k \geq 4$ and $\mathrm{diam}(D) \geq k + 2$ has a Hamiltonian path.

2. Main results

The following easy facts will be very useful in our proofs of main results.

Lemma 2.1 ([5]). *Let k be an integer with $k \geq 2$, D be a k-quasi-transitive digraph and $u, v \in V(D)$ such that there exists a (u, v)-path. Then each of the following holds:*

(1) *If $d(u, v) = k$, then $d(v, u) = 1$.*
(2) *If $d(u, v) = k + 1$, then $d(v, u) \leq k + 1$.*
(3) *Assume $d(u, v) = n \geq k+2$. If k is even, or k and n are both odd, then $d(v, u) = 1$; if k is odd and n is even, then $d(v, u) \leq 2$.*

Lemma 2.2 ([3]). *Let k be an even integer with $k \geq 2$ and D be a k-quasi-transitive digraph. Suppose that $P = x_0x_1 \ldots x_{k+2}$ is a shortest (x_0, x_{k+2})-path. Then each of the following holds:*

(a) $x_{k+2} \to \{x_0, x_1, \ldots, x_k\}$;
(b) *$x_{k+1} \to x_{k-i}$ for every even i such that $2 \leq i \leq k$.*

Lemma 2.3. *Let k be an even integer with $k \geq 2$ and D be a k-quasi-transitive digraph. Suppose that $P = x_0x_1 \ldots x_{k+2}$ is a shortest (x_0, x_{k+2})-path. Then $x_{k+1} \to x_{k-i}$ for every i such that $1 \leq i \leq k$.*

Proof. By Lemma 2.2(b), $x_{k+1} \to \{x_0, x_2, \ldots, x_{k-2}\}$. Below we prove that $x_{k+1} \to x_{k-i}$ by induction on odd i such that $1 \leq i \leq k - 1$.

By Lemma 2.2(a), $x_{k+2} \to \{x_0, x_1, \ldots, x_k\}$. Then $x_{k+1}x_{k+2}P[x_1, x_{k-1}]$ is a path of length k. By the definition of k-quasi-transitive digraphs, we have that $\overline{x_{k+1}x_{k-1}}$. This, together with the minimality of P, implies that $x_{k+1} \to x_{k-1}$.

For the inductive step, let us suppose that $x_{k+1} \to x_{k-i}$ for some odd i with $1 \leq i \leq k - 3$. By Lemma 2.1(1) and $d(x_0, x_k) = k$, we have $x_k \to x_0$. Then $x_{k+1}P[x_{k-i}, x_k]P[x_0, x_{k-(i+2)}]$ is a path of length k, which implies that $\overline{x_{k+1}x_{k-(i+2)}}$ and $x_{k+1} \to x_{k-(i+2)}$. Hence $x_{k+1} \to x_{k-i}$ for every odd i such that $1 \leq i \leq k - 1$. □

Lemma 2.4 ([2]). *Let D be a quasi-transitive digraph. Suppose that $P = x_0x_1 \ldots x_n$ is a shortest (x_0, x_n)-path. Then the subdigraph induced by $V(P)$ is a semicomplete digraph and $x_j \to x_i$ for $1 \leq i + 1 < j \leq n$, unless $n = 3$, in which case the arc between x_0 and x_n may be absent.*

Lemma 2.4 can be generalized to k-quasi-transitive digraphs with even k as follows.

Lemma 2.5. *Let k be an even integer with $k \geq 4$ and D be a k-quasi-transitive digraph. Suppose that $P = x_0x_1 \ldots x_n$ is a shortest (x_0, x_n)-path with $n \geq k + 2$ in D. Then $D[V(P)]$ is a semicomplete digraph and $x_j \to x_i$ for $1 \leq i + 1 < j \leq n$.*

Proof. Note that if $\overline{x_ix_j}$ and $1 \leq i + 1 < j \leq n$, then $x_j \to x_i$ since P is shortest. Hence we only need to show that $\overline{x_ix_j}$ for $1 \leq i + 1 < j \leq n$. We prove the result by induction on n.

First prove the case $n = k + 2$. By Lemma 2.2(a), $x_{k+2} \to \{x_0, x_1, \ldots, x_k\}$. By Lemma 2.3, $x_{k+1} \to \{x_0, x_1, \ldots, x_{k-1}\}$. Now we show $x_i \to \{x_0, x_1, \ldots, x_{i-2}\}$ for $2 \leq i \leq k$ by induction on i. For $i = 2$, the length of the path $x_2x_3 \ldots x_{k+1}x_0$ is k, which implies that $\overline{x_2x_0}$. For the inductive step, let us suppose that $x_i \to \{x_0, x_1, \ldots, x_{i-2}\}$ for $2 \leq i \leq k - 1$. Next we prove that