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Graphs whose characteristic and permanental polynomials have coefficients of the same magnitude

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ABSTRACT

Let $\phi(G,x)=\sum_{i=0}^{n} a_i x^{n-i}$ and $\pi(G,x)=\sum_{i=0}^{n} b_i x^{n-i}$ denote the characteristic polynomial and permanental polynomial of a graph G. In this paper, we consider the family $\mathscr{G}$ of graphs that have corresponding coefficients of the same magnitude, i.e., $|a_i| = |b_i|$ for $i = 0, 1, \ldots, n$. We prove that the graphs in this family are planar graphs. To characterize the structures of the considered graphs, we introduce the plane ear decomposition. With the help of the plane ear decomposition, we show the characterizations of the bipartite and non-bipartite graphs in this family.

1. Introduction

The graphs considered here are finite and simple with no loops and multiple edges. For a graph G, its vertex set is denoted by $V(G) = \{v_1, v_2, \ldots, v_n\}$. The *adjacency matrix* $A(G)$ of G is $(a_{ij})_{n\times n}$, where $a_{ij} = 1$ if there is an edge between v_i and v_j and $a_{ij} = 0$ otherwise. The *characteristic polynomial* of G is

$$\phi(G,x)=\det(xI-A(G))=\sum_{i=0}^{n} a_i x^{n-i},$$

where I is the $n\times n$ identity matrix. If two graphs have the same characteristic polynomial, then they are said to be *cospectral*. It is known that there are many cospectral graphs [5,9,11,17].

The *permanental polynomial* of G by definition is

$$\pi(G,x)=\operatorname{per}(xI-A(G))=\sum_{i=0}^{n} b_i x^{n-i},$$

where $\operatorname{per}(\cdot)$ denotes the *permanent* of a matrix.

To the best of our knowledge, the permanental polynomials were first introduced to differentiate cospectral graphs [13,14]. Later, the theory of permanental polynomials attracted much attention of chemical graph-theoreticians. The permanental polynomials of fullerenes and conjugated molecules were investigated in [6,12]. This study revealed that the coefficients and zeros of the permanental polynomial are related to the structure of molecules. For certain chemical graphs, the relations between the coefficients of the characteristic and permanental polynomials were discussed [7,10]. For general graphs, the coefficients of the characteristic and permanental polynomials were proved to be expressed by the structure of subgraphs [8,13].

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Let U_i denote the *basic figure* of a graph G, which is a subgraph on i vertices such that each component is a cycle or a single edge. Let $p(U_i)$ be the number of components of U_i and $c(U_i)$ the number of cycles of U_i. The coefficients of the characteristic and permanental polynomials are

$$a_i = \sum_{U_i \subset G} (-1)^{p(U_i)} 2^{c(U_i)}, \quad \text{for } 1 \le i \le n, \tag{1}$$

and

$$b_i = (-1)^i \sum_{U_i \subset G} 2^{c(U_i)}, \quad \text{for } 1 \le i \le n, \tag{2}$$

where the summation ranges over all basic figures U_i of G. In particular, $a_0 = b_0 = 1$.

Due to the fundamental formulae (1) and (2), it is proved that the coefficients of the characteristic and permanental polynomials of a tree have the same magnitude [4], i.e., $|a_i| = |b_i|$ for each i. As a generalization of this result, we shall consider further which types of graphs, in addition to trees, enjoy this property. For convenience, let $\mathscr{G}$ denote the family of graphs G with $|a_i| = |b_i|$ for each i. We will characterize the structure of graphs in $\mathscr{G}$ in this paper.

Throughout this paper, we use $|C|$ and $|P|$ to denote the lengths of a cycle C and a path P. Particularly, C_n denotes a cycle of length n. The *symmetric difference* $C_i \triangle C_j$ of two cycles C_i and C_j contains only the edges that are in exactly one of C_i or C_j.

An *ear* of a graph G is a path that is maximal with respect to internal vertices of degree 2 in G and is contained in a cycle of G. An *ear decomposition* of G is a decomposition $(C, P_1, \ldots, P_r)$ such that C is a cycle and P_i for $i \ge 1$ is an ear of $C \cup P_1 \cup \cdots \cup P_i$, denoted as $G = C + P_1 + P_2 + \cdots + P_r$. It was shown in [15] that "a graph G is 2-connected if and only if G has an ear decomposition starting with any cycle of G". Moreover, the end-vertices of each ear in this ear decomposition are different. In the following sections, the graphs in $\mathscr{G}$ will be characterized in terms of ear decompositions.

The rest of the paper is organized as follows: In Section 2, we show that the graphs in $\mathscr{G}$ are planar, and then introduce the plane ear decomposition to construct a plane graph. In Section 3, we present a sufficient and necessary condition for the bipartite graphs in $\mathscr{G}$. Moreover, we give a structure characterization of the 2-connected bipartite graphs in $\mathscr{G}$. In Section 4, after deriving the properties of the non-bipartite graphs in $\mathscr{G}$, we characterize the structure of such graphs.

2. The planarity of graphs belonging to $\mathscr{G}$

To investigate the graphs in $\mathscr{G}$, we test the planarity of these graphs, and construct the plane ear decomposition to characterize a plane graph.

2.1. The planarity of $G \in \mathscr{G}$

For the coefficients of the characteristic and permanental polynomials of a graph, an equivalent condition for $|a_i| = |b_i|$ is presented in [3], which is described as below.

Lemma 2.1 ([3]). *For a graph G and a fixed number i, $|a_i| = |b_i|$ holds if and only if all the basic figures U_i of G have the same parity of components.*

In this paper we use t and t_i to denote positive integers, and use s and s_i to denote non-negative integers. The *intersection condition* means that if C_{4t_i+1} intersects C_{4t_j+1} (resp. C_{4t_i+3} intersects C_{4t_j+3}) along a path l_1, then l_1 is of even length; if C_{4t_i+2} intersects C_{4t_j+1} (resp. C_{4t_i+2} intersects C_{4s_j+3}) along a path l_2, then l_2 is of odd length; if C_{4t_i+2} intersects C_{4t_j+2} along a path l_3, then l_3 is of odd length.

By Lemma 2.1, we can get the following technical lemma.

Lemma 2.2. *Let G be a connected graph in $\mathscr{G}$. Then it holds that*
(i) *G contains no subgraph that is isomorphic to C_{4t};*
(ii) *G contains no subgraph that is the disjoint union of C_{4t_i+1} and C_{4t_j+1} or the disjoint union of C_{4s_i+3} and C_{4s_j+3};*
(iii) *G contains no subgraph that has two cycles one of length $4t+1$ and the other of length $4s+3$;*
(iv) *the cycles in G satisfy the intersection condition.*

Proof. To prove (i), suppose to the contrary that G contains a cycle C_{4t}. We consider the two basic figures $U^1_{4t} = C_{4t}$ and U^2_{4t} the union of $2t$ independent edges. Both U^1_{4t} and U^2_{4t} are basic figures on $4t$ vertices. However, U^1_{4t} has one component and U^2_{4t} has $2t$ components. By Lemma 2.1, we have $|a_{4t}| \ne |b_{4t}|$. This contradicts $G \in \mathscr{G}$. Thus (i) holds.

Suppose to the contrary that there are two disjoint cycles C_{4t_i+1} and C_{4t_j+1} (resp. C_{4s_i+3} and C_{4s_j+3}) in G. Let the basic figure $U^1_{4(t_i+t_j)+2}$ (resp. $U^1_{4(s_i+s_j+1)+2}$) be two the cycles C_{4t_i+1} and C_{4t_j+1} (resp. C_{4s_i+3} and C_{4s_j+3}), and let the basic figure $U^2_{4(t_i+t_j)+2}$ (resp. $U^2_{4(s_i+s_j+1)+2}$) be a matching with $2(t_i + t_j) + 1$ (resp. $2(s_i + s_j + 1) + 1$) edges. We can see that $U^1_{4(t_i+t_j)+2}$ (resp. $U^1_{4(s_i+s_j+1)+2}$) has 2 components and $U^2_{4(t_i+t_j)+2}$ (resp. $U^2_{4(s_i+s_j+1)+2}$) has $2(t_i+t_j)+1$ (resp. $2(s_i+s_j+1)+1$) components. This contradicts Lemma 2.1. Thus (ii) is proved.