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Note

On realization graphs of degree sequences

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ABSTRACT

Given the degree sequence d of a graph, the realization graph of d is the graph having as its vertices the labeled realizations of d, with two vertices adjacent if one realization may be obtained from the other via an edge-switching operation. We describe a connection between Cartesian products in realization graphs and the canonical decomposition of degree sequences described by R.I. Tyshkevich and others. As applications, we characterize the degree sequences whose realization graphs are triangle-free graphs or hypercubes.

1. Introduction

Given the degree sequence d of a finite, simple graph, it is usually the case that d has several realizations, drawn from several distinct isomorphism classes. Understanding the structure of some or all of these realizations is a major focus in the study of degree sequences. If we wish to discuss the set of realizations as a whole, however, it is often useful to study an auxiliary graph, the so-called realization graph of d.

To describe the realization graph, we need some definitions. In this paper all graphs are simple and have finite, nonempty vertex sets. Given the set $[n] = \{1, \dots, n\}$ and the degree sequence $d = (d_1, \dots, d_n)$ of a simple graph (where we always assume that degree sequences are in descending order), a *realization of* d is a graph with vertex set $[n]$ such that the degree of vertex i is d_i for all $i \in [n]$.

Suppose that in a graph G there are four distinct vertices a, b, c, d such that ab and cd are edges of G and ad and bc are not. The 2-*switch* $\{ab, cd\} \rightrightarrows \{ad, bc\}$ is the operation of deleting edges ab and cd and adding edges ad and bc to G. (This operation has often also been called a *transfer* or *cycle exchange*.) Note that performing a 2-switch in a graph G results in a graph G' in which every vertex has the same degree as it had before; thus G' is another realization of the degree sequence of G.

The (2-*switch*) *realization graph* $\mathscr{G}(d)$ is the graph $(\mathscr{R}, \mathscr{E})$, where $\mathscr{R}$ is the set of realizations of d, and two vertices G, G' of $\mathscr{R}$ are adjacent if and only if performing some 2-switch changes G into G'. Since the operation of undoing a 2-switch is itself a 2-switch, $\mathscr{G}(d)$ may be thought of as an undirected graph. We provide examples of $\mathscr{G}(d)$ for a few specific degree sequences in the next section.

It is unclear from the literature where the realization graph as we have defined it first appeared, though related notions have appeared in multiple contexts. For instance, the interchange graph of a score sequence, as introduced by Brualdi and Li [5], takes as its vertices the tournaments having a given score sequence; edges join tournaments that differ only on the orientation of a single directed triangle. The paper [5] shows that this graph is a regular connected bipartite graph; other results appear in [9] and [29].

Many authors have studied another interchange graph, also introduced by Brualdi [4], which has as its vertices the (0, 1) matrices having prescribed row and column sums. Here edges join vertices that differ by a simple switch of entries; interpreting each matrix as a biadjacency matrix of a bipartite graph, these switches correspond to 2-switches that preserve the

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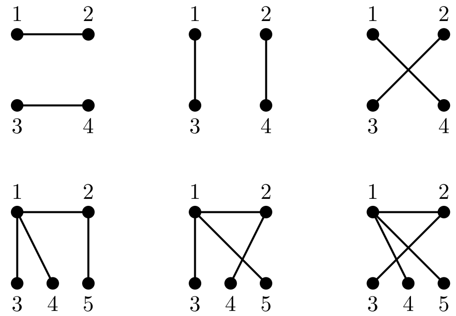

Fig. 1. Realizations of (1, 1, 1, 1) and (3, 2, 1, 1, 1).

partite sets. The papers [6,7,10,17,18,24–26,33,34] include results on such properties of this interchange graph as its diameter and lengths of its cycles. As Arikati and Peled pointed out in [1], each of these interchange graphs arises as the realization graph $\mathscr{G}(d)$ for the degree sequence d of a split graph obtained by adding edges to any of the associated bipartite graphs to make one of the partite sets a clique.

The realization graph $\mathscr{G}(d)$ seems to have attracted less attention in its general setting, where d may be the degree sequence of a non-split graph. The best-known result on $\mathscr{G}(d)$ is that it is a connected graph for any degree sequence d; this is a consequence of a theorem of Fulkerson, Hoffman, and McAndrew [15] (Petersen proved the same result for regular degree sequences in [23]; see also Senior [28]).

A major question of study, proposed in [4] and as yet unresolved, is whether $\mathscr{G}(d)$ always has a Hamiltonian cycle (or is K_2). Results on interchange graphs, such as those in [18,34], yield partial results for the degree sequences of split graphs. The paper [1] shows that $\mathscr{G}(d)$ is Hamiltonian if d has threshold gap 1.

In this paper we provide a structure theorem for the realization graph $\mathscr{G}(d)$ and comment on another class of degree sequences d for which $\mathscr{G}(d)$ is Hamiltonian. After beginning with some important examples of realization graphs and recalling some definitions in Section 2, we show in Section 3 that a certain structural decomposition of degree sequences due to Tyshkevich [30,32], called the *canonical decomposition*, allows us to express the corresponding realization graphs as Cartesian products of smaller realization graphs.

This structural result then allows us in Section 4 to characterize the realization graphs that are triangle-free (and equivalently, the realization graphs that are bipartite); these are precisely the realization graphs of degree sequences of pseudo-split matrogenic graphs. We also show that the degree sequences whose realization graphs are hypercubes are precisely the degree sequences of split P_4-reducible graphs. (All terms will be defined later.)

Since the Hamiltonicity of a Cartesian product follows from the Hamiltonicity of its factors, the canonical decomposition of a degree sequence may be used as an aid in characterizing d for which $\mathscr{G}(d)$ is Hamiltonian. As an illustration, we conclude Section 4 by using our results to show that all triangle-free realization graphs are Hamiltonian.

Throughout the paper we denote the vertex set of a graph G by $V(G)$. For an integer $n \geq 1$, we use K_n, P_n, and C_n to denote the complete graph, path, and cycle on n vertices, respectively; the complete bipartite graph with partite sets of sizes m and n is $K_{m,n}$. For a family $\mathcal{F}$ of graphs, a graph G is $\mathcal{F}$*-free* if G contains no element of $\mathcal{F}$ as an induced subgraph.

2. Preliminaries

In this section we lay some groundwork, beginning with examples of specific realization graphs that will be important in Section 4.

Example 1. As shown in [1], when d is (1, 1, 1, 1) or (3, 2, 1, 1, 1), $\mathscr{G}(d)$ is isomorphic to the triangle K_3. Fig. 1 shows the realizations of both of these sequences; it is easy to verify that from any realization of either of these sequences d, either of the other two realizations may be obtained via a single 2-switch.

Example 2. When $d = (2, 2, 2, 2, 2)$, the set $\mathscr{R}$ of realizations of d contains 12 graphs, each isomorphic to C_5. Each realization allows exactly 5 distinct 2-switches. Performing one of these 2-switches transforms the graph into a cycle that visits the vertices in the same order as before, save that two consecutive vertices on the cycle exchange places. (See Fig. 2 for an illustration.) As observed in [12], the realization graph $\mathscr{G}(d)$ is then isomorphic to $K_{6,6}$ minus a perfect matching.

Example 3. Suppose d is degree sequence $(k, \dots, k, 1, \dots, 1)$ consisting of k copies of k and k ones (where $k \geq 1$). Every realization of d is a net, where a (k-)*net* is a graph with vertex set $\{a_1, \dots, a_k, b_1, \dots, b_k\}$ and edge set $\{a_ib_i : i \in [k]\} \cup \{b_ib_j : i, j \in [k]\}$ for some $k \geq 1$. A graph G is a *net-complement* if its complement is a net.

Observe that in any k-net, every 2-switch involves edges a_ib_i, a_jb_j and non-edges a_ib_j, a_jb_i for some $i, j \in [k]$, and every pair of distinct elements i, j from $[k]$ yields such a 2-switch. Performing one of these 2-switches has the same effect on a realization as exchanging the names of vertices a_i and a_j.