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Note

A generalization of a cyclotomic family of partial difference sets given by Fernández-Alcober, Kwashira and Martínez

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ABSTRACT

In this paper we present a new cyclotomic family of partial difference sets, which includes a family presented in Fernández-Alcober et al. (2010). The argument rests on a general procedure for constructing cyclotomic difference sets or partial difference sets in Galois domains due to Ott (2015).

1. Introduction

For all notions concerning partial difference sets we refer to Ma [9]. A partial difference set is simple to define. A subset Δ of a finite group G with the property such that

$$1 \notin \Delta \quad \text{and} \quad \Delta^{[-1]} = \{x^{-1} \mid x \in \Delta\} = \Delta$$

is said to be a partial difference set of G, if for some fixed natural numbers λ and μ, every element $g \neq 1$ of G admits exactly λ resp. μ representations of the form

$$g = xy^{-1}, \ (x, y) \in \Delta \times \Delta,$$

if $g \in \Delta$ resp. $g \notin \Delta$. If Δ is a partial difference set in G, the group G becomes a strongly regular graph by introducing the adjacency relation $a \sim b \Leftrightarrow ab^{-1} \in \Delta$. But we remark that we allow $\mu = 0$, in contrast to some definitions in literature.

We shall construct a new family of cyclotomic partial difference sets, which includes the family of Fernández-Alcober, Kwashira and Martínez given in [7], Proposition 4.3.

Let K_1, K_2 be finite fields with q_1, q_2 elements. We ask for criteria which will guarantee that a particular subgroup U of $K_1^\star$ resp. $K_1^\star \times K_2^\star$ yields a partial difference set of the form

$$\Delta = U \text{ resp. } \Delta = K_1^\star \cup U$$

in

$$\mathbb{D} = K_1 \text{ resp. } \mathbb{D} = K_1 \oplus K_2,$$

where we identify $K_1^\star$ with $K_1^\star \times \{0\}$, if necessary.

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Our main argument for constructing cyclotomic partial difference sets is based on a general procedure due to Ott [10], Theorems 14 and 20. But in this paper we need only the following simpler version of these theorems:

Theorem 1. *Let χ be a character of $K_1^\star$ having order $1 < d < q_1 - 1$ such that $\chi(-1) = 1$. Then the kernel $U = \ker\chi$ is a partial difference set of K_1 with parameters λ, μ, if and only if*

$$d(\mu - \lambda) = \sum_{j=0}^{d-1} J(\chi^i, \chi^j)$$

for all $1 \le i \le d - 1$.

Here $J(\chi^i, \chi^j)$ denotes the Jacobi sum of χ^i and χ^j. Definitions and results on Jacobi sums can be found for instance in Berndt, Evans, Williams web or Lemmermeyer [8]. But it is important to remark that our definition of a Jacobi sum rests on [8], which differs in a sign from the definition given in [3].

Theorem 2. *Let $\chi = \chi_1\chi_2$ be a character of $K_1^\star \times K_2^\star$, such that both χ_1 and χ_2 have order $1 < d < q_1 - 1, q_2 - 1$ and $\chi(-1) = 1$. Then $K_1^\star \cup \ker\chi$ is a partial difference set of $K_1 \oplus K_2$ with parameters*

$$v = q_1q_2, \quad k = \frac{(q_1-1)(q_2-1)}{d} + q_1 - 1, \quad \lambda = \mu_1 + q_1 - 2, \quad \mu = \lambda_1 + 2r\frac{q_1-1}{d},$$

where

$$\mu_1 = \frac{q_2-1}{d}\left(\frac{q_1-1}{d} - 1\right) \quad \textit{and} \quad \lambda_1 = \frac{q_1-1}{d}\left(\frac{q_2-1}{d} - 1\right),$$

if and only if

$$d(\lambda - \mu + 2) = d\left(q_1 - \frac{q_1+q_2-2}{d}\right) = \sum_{j=0}^{d-1} J(\chi^i, \chi^j)$$

for all $1 \le i \le d - 1$.

Here

$$J(\chi^i, \chi^j) = J(\chi_1^i, \chi_1^j)J(\chi_2^i, \chi_2^j)$$

is the product of (classical) Jacobi sums.

Now let K be a finite field with q elements and let K_0 be a proper subfield of K with q_0 elements. Clearly, K is the intersection of

$$K_1 = \mathbb{F}_{q^a} \text{ and } K_2 = \mathbb{F}_{q^{a+1}}, \quad a \ge 1.$$

The construction of U is based on the fact that $q^a - 1, q^{a+1} - 1 \equiv 0 \pmod{q-1}$. Our result is the following:

Theorem 3. *Choose a multiplicative character ξ of the field K of order $d = (q-1)/(q_0-1)$ and let χ_1, χ_2 be the lifts of ξ^{-1}, ξ to K_1, K_2, respectively. Set $\chi = \chi_1\chi_2$ and let*

$$U = U(\chi) = \ker\chi = \{(x, y) \mid x, y \ne 0,\ \chi_1(x)\chi_2(y) = 1\}.$$

Then

$$\Delta = K_1^\star \cup U$$

is a partial difference set with parameters

$$v = q^{2a+1}, \quad k = \frac{(q^a-1)(q-q_0+q^{a+1}(q_0-1))}{q-1}$$

$$\lambda = q^a - \frac{(q_0-1)\left(q^{a+1}-1\right)\left(-(q_0-1)q^a + q + q_0 - 2\right)}{(q-1)^2} - 2, \quad \mu = \frac{k}{d}.$$

2. Proof of the theorem

We preserve the notation introduced in above section. Furthermore, set

$$\mathcal{H} = \langle\chi\rangle.$$