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# On endomorphisms of alternating forms graph☆

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ABSTRACT

A graph $G$ is a pseudo-core if every endomorphism of $G$ is either an automorphism or a colouring. Let $\mathbb{F}_q$ be the finite field with $q$ elements and let $\mathrm{Alt}(m, q)(m \geq 4)$ be the alternating forms graph on the vector space $\mathbb{F}_q^m$. We prove that $\mathrm{Alt}(m, q)$ is a pseudo-core. Moreover, if $m$ is odd, then $\mathrm{Alt}(m, q)$ is a core. If both $m$ and $q$ are even, then $\mathrm{Alt}(m, q)$ is not a core.



## 1. Introduction

Throughout this paper, all graphs are *simple* [6] and finite. Let $\mathbb{F}_q$ be the finite field with $q$ elements where $q$ is a power of a prime, and let $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$. Denote by $\mathbb{F}_q^{m\times n}$ the set of $m \times n$ matrices over $\mathbb{F}_q$ and $\mathbb{F}_q^m = \mathbb{F}_q^{1\times m}$. Let $|X|$ be the cardinal number of a set $X$. In a graph $G$, the *distance* $d(x, y)$ between vertices $x$ and $y$ is the number of edges of a shortest path from $x$ to $y$. We write $x \sim y$ (resp. $x \nsim y$) if vertices $x$ and $y$ are adjacent (resp. nonadjacent). Let $V(G)$ be the vertex set of a graph $G$.

Assume that $G, H$ are graphs and $\varphi : V(G) \to V(H)$ is a map. If $\varphi(x) \sim \varphi(y)$ whenever $x \sim y$, then $\varphi$ is called a *homomorphism*. If $\varphi$ is bijective and $x \sim y \Leftrightarrow \varphi(x) \sim \varphi(y)$, then $\varphi$ is an *isomorphism*. A homomorphism (resp. isomorphism) from $G$ to itself is called an *endomorphism* (resp. *automorphism*) of $G$. If $H$ is the complete graph $K_r$, then $\varphi$ is called a *r-colouring* (*colouring* for short) of $G$. The *chromatic number* $\chi(G)$ of $G$ is the least value $k$ for which $G$ can be $k$-coloured.

Recall that a *clique* of a graph $G$ is a subgraph of $G$ that is complete; a *maximal clique* of $G$ is a clique $C$ of $G$ such that there is no clique of $G$ which properly contains $C$ as a subset; a *maximum clique* of $G$ is a clique of $G$ which has maximum cardinality. The *clique number* $\omega(G)$ of a graph $G$ is the number of vertices in a maximum clique. In the language of geometry of matrices [20,9,11], we use the term *maximal set* to indicate a *maximal clique*.

A graph $G$ is a *core* [6] if every endomorphism of $G$ is an automorphism. A subgraph $\Gamma$ of a graph $G$ is a *core of* $G$ [6] if it is a core and there exists some homomorphism from $G$ to $\Gamma$. A graph $G$ is called *core-complete* [7] if it is a core or its core is complete.

A graph $G$ is called a *pseudo-core* if every endomorphism of $G$ is either an automorphism or a colouring. It is easy to see that a graph $G$ is a pseudo-core if and only if every endomorphism of $G$ is either an automorphism or its range is a maximum clique. Thus, if a graph $G$ is a pseudo-core but not a core, then there is an endomorphism of $G$ such that its range is a maximum clique. Every core is a pseudo-core. Every pseudo-core is core-complete but not vice versa.

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For a graph $G$, an *independent set* of $G$ is a subset of vertices such that no two vertices are adjacent. A *largest independent set* of $G$ is an independent set of maximum cardinality. The *independence number* $\alpha(G)$ of $G$ is the number of vertices in a *largest independent set* of $G$ [6].

For an $m \times m$ ($m \geq 2$) matrix $A$ over $\mathbb{F}_q$, let ${}^tA$ be the *transpose matrix* of $A$. If ${}^tA = -A$ with zero diagonal elements, then $A$ is called an *alternate matrix*. $A$ is an $m \times m$ alternate matrix if and only if $xA^tx = 0$ for all $x \in \mathbb{F}_q^m$ (cf. [20, p.39]). If $q$ is odd, then $A$ is alternate if and only if $A$ is skew-symmetric (or $A = -{}^tA$). However, if $q$ is even, then a skew-symmetric matrix may not be alternate because $a = -a$ for all $a \in \mathbb{F}_q^*$. The rank of any alternate matrix is even (cf. [20, Proposition 1.34] or [22, Theorem 11.11]).

Let $\mathcal{K}_m(\mathbb{F}_q)$ ($\mathcal{K}_m$ for short) be the set of $m \times m$ ($m \geq 2$) alternate matrices over $\mathbb{F}_q$. The *alternating forms graph* $\mathrm{Alt}(m, q)$ has the vertex set $\mathcal{K}_m(\mathbb{F}_q)$, and two vertices $A$ and $B$ are adjacent if $\mathrm{rank}(A - B) = 2$. By [2, Theorem 9.5.3], every alternating forms graph is *distance-transitive*. By the geometry of alternate matrices (cf. [20, Prop. 4.5] or below formula (1)), every alternating forms graph is connected. By [7, Corollary 4.2], any alternating forms graph is core-complete. The alternating forms graph plays an important role in geometry, graph theory, group theory, association schemes and coding theory [21,20,23,11].

Recently, literatures [15,14,10] characterized the endomorphisms for *symmetric bilinear forms graphs*, *hermitian forms graphs* and *bilinear forms graphs* over a finite field, respectively. However, the characterization of the endomorphisms of *Alt*$(m, q)$ remains open. The main result of this paper is as follows.

**Theorem 1.1.** *Let $m$ be an integer $\geq 4$, and let $G = \mathrm{Alt}(m, q)$. Then $G$ is a pseudo-core. Moreover, if $m$ is odd, then $G$ is a core. If $m$ is even, then $G$ is not a core if and only if its independence number is $q^{(m-1)(m-2)/2}$. In particular, if both $m$ and $q$ are even, then $G$ is not a core.*

The paper is organized as follows. In Section 2, we characterize maximal (maximum) cliques. In Section 3, we discuss the core, independence number and chromatic number of $\mathrm{Alt}(m, q)$. In Section 4, we prove that if an endomorphism $\varphi$ of $\mathrm{Alt}(m, q)$ maps two distinct type one maximal cliques (their intersection is nonempty) onto a maximum clique, then $\mathrm{Im}(\varphi)$ is a maximum clique. Finally, we will prove Theorem 1.1.

## 2. Maximal cliques and maximum cliques of Alt$(m, q)$

In order to prove Theorem 1.1, we need to discuss properties of maximal cliques and maximum cliques of $\mathrm{Alt}(m, q)$.

Recall that every alternate matrix has even rank. If $m = 2, 3$, then $\mathrm{rank}(A - B) = 2$ for any two distinct $A, B \in \mathcal{K}_m$; thus $\mathrm{Alt}(m, q)$ is a clique and it is a core. From now on, we always assume $m \geq 4$ for $G = \mathrm{Alt}(m, q)$ in our discussion unless specified otherwise.

In $\mathrm{Alt}(m, q)$, it is well-known that (cf. [20, Prop. 4.5])

$$d(A, B) = \frac{1}{2}\mathrm{rank}(A - B), \quad A, B \in \mathcal{K}_m. \tag{1}$$

Denote by $GL_n(\mathbb{F}_q)$ the set of all $n \times n$ invertible matrices over $\mathbb{F}_q$. Let $E_{ij}^{n\times n}$ ($E_{ij}$ for short) be the $n \times n$ matrix whose $(i, j)$-entry is 1 and all other entries are 0's, and let $D_{ij} = E_{ij} - E_{ji}$ where $i \neq j$. Let $I_r$ be the $r \times r$ identity matrix. Denote by $0_{m,n}$ the $m \times n$ zero matrix (0 for short), and $0_n = 0_{n,n}$. Let $\mathrm{diag}(A_1, \ldots, A_k)$ be a block diagonal matrix over $\mathbb{F}_q$. We denote by $\mathbb{F}_qD_{ij}$, $\mathcal{M}_i$ and $\mathcal{N}_j$ the subsets of $\mathcal{K}_m$ which are defined by $\mathbb{F}_qD_{ij} = \left\{xD_{ij} : x \in \mathbb{F}_q\right\}$,

$$\mathcal{M}_i = \left\{\sum_{j=1, j\neq i}^{m} x_jD_{ij} : x_j \in \mathbb{F}_q\right\} \quad (i = 1, \ldots, m)$$

and

$$\mathcal{N}_j = \left\{xD_{12} + yD_{1j} + zD_{2j} : x, y, z \in \mathbb{F}_q\right\} \quad (j = 3, \ldots, m).$$

Two $n \times n$ matrices $A$ and $B$ over $\mathbb{F}_q$ are said to be *congruent*, denoted by $A \approx B$, if there exists a $P \in GL_n(\mathbb{F}_q)$ such that ${}^tPAP = B$. For a subset $\mathcal{S}$ of $\mathcal{K}_m$, we write ${}^tP\mathcal{S}P + A = \{{}^tPXP + A : X \in \mathcal{S}\}$, where $A \in \mathcal{K}_m$ and $P \in GL_m(\mathbb{F}_q)$. Put $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $J^{(r)} = \mathrm{diag}(J, \ldots, J) \in \mathcal{K}_{2r}$.

**Lemma 2.1** (*[20, Proposition 1.34] or [22, Theorem 11.11]*)**.** *If $A \in \mathcal{K}_m$ ($m \geq 2$), then the rank of $A$ is necessarily even. Further, if $A$ is of* rank $2r$, *then* $A \approx \mathrm{diag}(J^{(r)}, 0)$ *and*

$$A \approx \begin{pmatrix} 0 & I_r & 0 \\ -I_r & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2}$$

For convenience, if $\mathcal{M}$ is a maximal clique (maximum clique) of a graph $G$, then we regard that $\mathcal{M}$ and $V(\mathcal{M})$ are the same.