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Triple systems and binary operations

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ABSTRACT

It is well known that given a Steiner triple system (STS) one can define a binary operation $*$ upon its base set by assigning $x * x = x$ for all x and $x * y = z$, where z is the third point in the block containing the pair $\{x, y\}$. The same can be done for Mendelsohn triple systems (MTSs) as well as hybrid triple systems (HTSs), where (x, y) is considered to be ordered. In the case of STSs and MTSs, the operation is a quasigroup, however this is not necessarily the case for HTSs. In this paper we study the binary operation induced by HTSs. It turns out that each such operation $*$ satisfies

$$y \in \{x * (x * y), (x * y) * x\} \quad \text{and} \quad y \in \{(y * x) * x, x * (y * x)\}$$

for all x and y from the base set. We call every binary operation that fulfils this condition *hybridly symmetric*.

Not all idempotent hybridly symmetric operations can be obtained from HTSs. We show that these operations correspond to decompositions of a complete digraph into certain digraphs on three vertices. However, an idempotent hybridly symmetric quasigroup always comes from an HTS. The corresponding HTS is then called a *latin HTS* (LHTS). The core of this paper is the characterization of LHTSs and the description of their existence spectrum.

1. Introduction

Consider an ordered pair $(X, \mathcal{B})$, where X is a set of points and $\mathcal{B}$ is a decomposition of the complete digraph on X into *cyclic triples* (a, b, c) and *transitive triples* $\langle a, b, c\rangle$. The cyclic triple (a, b, c) consists of arrows (i.e. directed edges) (a, b), (b, c) and (c, a), while a transitive triple $\langle a, b, c\rangle$ carries (a, b), (b, c) and (a, c). If all triples in $\mathcal{B}$ are cyclic, then $(X, \mathcal{B})$ is called a *Mendelsohn triple system* (MTS). If all triples in $\mathcal{B}$ are transitive, then it is called a *directed triple system* (DTS). If we allow both cyclic and transitive triples to occur in $\mathcal{B}$, then the term *hybrid triple system* (HTS) is used, following Colbourn, Pulleyblank and Rosa [5]. However, the concept of an HTS seems to have appeared earlier (under a different name) in an article [18] of Lindner and Street, and later, independently, in [22].

Each HTS induces a binary operation, say $*$, upon its base set. For a cyclic triple (a, b, c) set $a * b = c$, $b * c = a$ and $c * a = b$. For a transitive triple $\langle a, b, c\rangle$ set $a * b = c$, $b * c = a$ and $a * c = b$. The *induced operation* $*$ is assumed to be idempotent, i.e. $a * a = a$ holds for every a.

It is easy to see that the binary operation $*$ is induced by an MTS if and only if it is semisymmetric (i.e. $x*(y*x) = y$ for all x and y). If a binary operation satisfies the semisymmetric law, then it is a quasigroup, and, as is well known [1, Remark 2.12],

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there is a one-to-one correspondence between MTS(n)s and idempotent semisymmetric quasigroups of order n. In [9,8] we were concerned with a combinatorial and an algebraic description of those DTSs that yield a quasigroup. In this paper we give a similar description for HTSs that yield a quasigroup (Theorems 5.3 and 5.4). We call any such HTS a *latin hybrid triple system* (LHTS). The corresponding quasigroups can be described as idempotent hybridly symmetric quasigroups, i.e. idempotent quasigroups that fulfil

$$y \in \{x * (x * y), (x * y) * x\} \quad \text{and} \quad y \in \{(y * x) * x, x * (y * x)\}$$

for all x and y of the base set.

In fact, every binary operation $*$ that is induced by an HTS (regardless of whether it is a quasigroup or not) is hybridly symmetric. However, not all idempotent hybridly symmetric operations can be obtained from HTSs. We shall see that such an operation corresponds to an HTS if and only if it satisfies an additional condition (Proposition 3.5). It is then natural to ask if there is a combinatorial interpretation for idempotent hybridly symmetric operations. The answer is positive and points to the decompositions of K_n^* into digraphs on three vertices as studied by Hartman and Mendelsohn [14]. There are exactly seven digraphs on three vertices which have the property that for any two distinct vertices a and b at least one of the arrows (a, b) and (b, a) is an edge of the digraph. It is possible to define $*$ for these digraphs similarly as above and, as we prove in Section 3, decompositions of K_n^* into these seven digraphs correspond to idempotent hybridly symmetric operations (Theorem 3.3). The correspondence is not one-to-one, since if the decomposition contains a pair of triples with the same vertex set, say $\langle a, b, c\rangle$ and $\langle c, b, a\rangle$, then these can be replaced by a different pair of triples, say $\langle b, a, c\rangle$ and $\langle c, a, b\rangle$, however both systems induce the same binary operation. To get a one-to-one correspondence the notion of a coarse decomposition is needed (cf. Section 3).

The algebraic descriptions that are introduced in this paper have, admittedly, certain disadvantages: they cannot be used for decompositions into λ-fold digraphs if $\lambda > 1$, and they cannot be used to study mixed triple systems [10]. But there are also advantages. One of them is the ease with which examples can be produced by standard tools that generate first-order models. We exploit this feature in Section 4 where we give the number of isomorphism types for uniform systems that are based upon each of the seven digraphs on three elements, for a base set of 10 and less elements (with one exception). Also, in Section 6, we count the number of isomorphism types for proper LHTSs, again up to a base set of 10 elements (see also the Appendix). In doing so, we have used the algebraic description of LHTSs given in Theorem 5.4, which states that LHTSs correspond to idempotent binary operations $*$ satisfying

$$y = x * (x * y) = (y * x) * x \quad \text{or} \quad y = (x * y) * x = x * (y * x)$$

for all x and y of the base set.

In Section 7 we prove that a proper LHTS of order n exists if and only if $n \equiv 0$ or $1 (\mathrm{mod}\ 3)$ and $n \geq 9$. The proof is constructive and yields LHTSs which are balanced in the sense that asymptotically half of the triples are transitive and the other half are cyclic.

2. Symmetric operations (totally, left, right, middle, hybridly)

The fact that Steiner triple systems (STS) are equivalent to totally symmetric idempotent quasigroups is often mentioned without realizing where the notion of total symmetricity comes from.

A quasigroup operation $*$ is said to be *totally symmetric* if $x * y = y * x = x/y = y/x = x \setminus y = y \setminus x$ for all x and y. Here $\setminus$ and $/$ stand for the left and right division in the quasigroup. It is immediately clear that this class of quasigroups is determined already by the laws $x * y = y * x = y/x$. Instead of $y * x = y/x$ one can write $(y * x) * x = y$, and hence totally symmetric quasigroups can be fully described by

$$(y * x) * x = y \quad \text{and} \quad x * y = y * x.$$

A binary operation fulfilling these two laws is necessarily a quasigroup, and so there is no difference between totally symmetric (binary) operations and totally symmetric quasigroups. If the operation is idempotent (i.e. $x*x = x$ for every element x), then the sets $\{x, y, x * y\}$, $x \neq y$, determine an STS. The converse holds as well. The correspondence between STSs and totally symmetric loops (quasigroups with a unit) is also widely known.

The notion of totally symmetric quasigroups is natural. It is not really about a symmetry, but about the coincidence of all six parastrophic operations. The equational description of totally symmetric quasigroup seems to have been the main reason why the law $(y * x) * x = y$ has been called *right symmetric* and the law $x * (x * y) = y$ *left symmetric*. Admittedly, these terms are little illuminating in themselves. Nevertheless, they are widely accepted and in such circumstances it seems reasonable to call the law $x * (y * x) = y$ *middle symmetric*. However, this law is nearly always called semisymmetric since it represents a half of the characterization of totally symmetric quasigroups by $x * (y * x) = y$ and $x * y = y * x$.

Lemma 2.1. *Let $*$ be a binary operation upon a set X. If $x * (y * x) = y$ for all $x, y \in X$, then $(x * y) * x = y$ for all $x, y \in X$ as well.*

Proof. We have $(x * y) * x = (x * y) * (y * (x * y)) = y$. $\square$

The term *hybridly symmetric* introduced above thus expresses the fact that for any (x, y) we get an instance of left or middle symmetric law, and an instance of right or middle symmetric law.