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Note

On the number of overpartitions into odd parts

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ABSTRACT

Let $\overline{po}(n)$ be the number of overpartitions of n into odd parts. We prove an identity of $\overline{po}(n)$ and establish many explicit Ramanujan-like congruences for $\overline{po}(n)$ modulo 32 and 64.

1. Introduction

An overpartition of a positive integer n is a partition of n in which the first occurrence of each part can be overlined. For example, there are fourteen overpartitions of 4:

$$4,\quad \overline{4},\quad 3+1,\quad \overline{3}+1,\quad 3+\overline{1},\quad \overline{3}+\overline{1},\quad 2+2\quad \overline{2}+2,\quad 2+1+1,$$
$$\overline{2}+1+1,\quad 2+\overline{1}+1,\quad \overline{2}+\overline{1}+1,\quad 1+1+1+1,\quad \overline{1}+1+1+1.$$

We denote the number of overpartitions of n by $\overline{p}(n)$. The generating function for $\overline{p}(n)$ is

$$\sum_{n=0}^{\infty}\overline{p}(n)q^n=\prod_{n=1}^{\infty}\frac{(1+q^n)}{(1-q^n)}=\prod_{n=1}^{\infty}\frac{(1-q^{2n})}{(1-q^n)^2}=1+2q+4q^2+8q^3+14q^4+\cdots.$$

Overpartitions have been extensively studied, and they possess many analogous properties to ordinary partitions, see, for example, [3,4,7,6,11–14].

In this note, we consider the overpartitions of n into odd parts. We denote by $\overline{po}(n)$ the number of such partitions. It is clear that

$$\sum_{n=0}^{\infty}\overline{po}(n)q^n=\prod_{n=1}^{\infty}\frac{1+q^{2n+1}}{1-q^{2n+1}}=\prod_{n=1}^{\infty}\frac{(1-q^{2n})^3}{(1-q^n)^2(1-q^{4n})}=1+2q+2q^2+4q^3+\cdots.$$

This generating function has appeared in the works of Ardonne, Kedem, and Stone [1], Bessenrodt [2], Santos and Sills [16]. Recently, arithmetic properties of $\overline{po}(n)$ were considered by Hirschhorn and Sellers [8]. They established several Ramanujan-like congruences satisfied by $\overline{po}(n)$ and some easily-stated characterizations of $\overline{po}(n)$ modulo 4 and 8. For example, they proved that for all $n\geq 0,\ \alpha>0$,

$$\overline{po}(2^{\alpha}(8n+5))\equiv 0 \pmod 8),$$
$$\overline{po}(8n+7)\equiv 0 \pmod{16}.$$

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It is a natural question to find explicit congruences for $\overline{po}(n)$ modulo higher powers of 2. The aim of this note is to establish some explicit congruences for $\overline{po}(n)$ modulo 32 and 64. Our main result is Theorem 2 below, which indeed yields infinitely many Ramanujan-like congruences for $\overline{po}(n)$.

We first give a new identity of $\overline{po}(n)$. For brevity we shall use the following notation: for positive integers a, b,

$$E(a)^b := \prod_{n=1}^{\infty}(1-q^{an})^b.$$

Theorem 1. *We have*

$$\sum_{n=0}^{\infty}\overline{po}(16n+14)q^n = 112\frac{E(2)^{27}}{E(1)^{25}E(4)^2} + 256q\frac{E(2)^3E(4)^{14}}{E(1)^{17}}.$$

Theorem 1 implies $\overline{po}(16n+14) \equiv 0 \pmod{16}$ clearly. Using elementary theory of modular forms, we can extend this congruence to modulo 32 and 64.

Theorem 2. *Let $t \geq 0$ be an integer and $p \equiv 1 \pmod 8$ be a prime. Then for all nonnegative integers n with $n \not\equiv -\frac{7}{8} \pmod p$,*

$$\overline{po}(16p^{2t+1}n + 16\lambda_{p,t} + 14) \equiv 0 \pmod{32},$$
$$\overline{po}(16p^{4t+3}n + 16\delta_{p,t} + 14) \equiv 0 \pmod{64},$$

where $\lambda_{p,t} = \frac{7(p^{2t+1}-1)}{8}$ and $\delta_{p,t} = \frac{7(p^{4t+3}-1)}{8}$. Suppose that $p_1, p_2 \equiv 1 \pmod 8$ are two distinct primes. Then for all nonnegative integers n satisfying $n \not\equiv -\frac{7}{8} \pmod{p_1}$ and $n \not\equiv -\frac{7}{8} \pmod{p_2}$,

$$\overline{po}(16p_1p_2n + 16\delta_{p_1p_2} + 14) \equiv 0 \pmod{64},$$

where $\delta_{p_1p_2} = \frac{7(p_1p_2-1)}{8}$.

Example. To illustrate Theorem 2, we let $p = 17$. Then $\lambda_{17,0} = 14$, $\delta_{17,0} = 4298$. Theorem 2 states that

$$\overline{po}(272n + 238) \equiv 0 \pmod{32},$$
$$\overline{po}(16 \cdot 17^3 n + 68782) \equiv 0 \pmod{64}$$

for all $n \not\equiv 14 \pmod{17}$. Let $p_1 = 17$, $p_2 = 41$. Then $\delta_{17,41} = 609$, and for all $n \not\equiv 14 \pmod{17}$ and $n \not\equiv 35 \pmod{41}$,

$$\overline{po}(16 \cdot 17 \cdot 41n + 9758) \equiv 0 \pmod{64}.$$

2. Proof of Theorem 1

We need the following two lemmas.

Lemma 1. *Let*

$$\phi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{E(2)^5}{E(1)^2E(4)^2},$$
$$\psi(q) := \sum_{n=-\infty}^{\infty} q^{2n^2-n} = \frac{E(2)^2}{E(1)}.$$

Then

$$\phi(q) = \phi(q^4) + 2q\psi(q^8).$$

Lemma 2. *We have*

$$\overline{P}(q) = \frac{E(2)}{E(1)^2} = \frac{E(8)^5}{E(2)^4E(16)^2} + 2q\frac{E(4)^2E(16)^2}{E(2)^4E(8)}.$$

The proof of Lemma 1 is obvious. Lemma 2 is the 2-dissection of $\overline{P}(q)$ which can be found in Theorem 1 of [7].

Proof of Theorem 1. We observe that

$$\sum_{n=0}^{\infty}\overline{po}(n)q^n = \frac{E(2)^3}{E(1)^2E(4)} = \phi(q)\overline{P}(q^2).$$