Discrete Mathematics 325 (2014) 77–84

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Counting edge-Kempe-equivalence classes for 3-edge-colored cubic graphs

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ARTICLE INFO

Article history:
Received 3 September 2012
Received in revised form 12 October 2013
Accepted 17 February 2014
Available online 6 March 2014

Keywords:
Edge-coloring
Kempe chains
Coloring graphs
Cubic graphs

ABSTRACT

Two n-edge colorings of a graph are *edge-Kempe equivalent* if one can be obtained from the other by a series of edge-Kempe switches. In this work we show every planar bipartite cubic graph has exactly one edge-Kempe equivalence class, when $3 = \chi'(G)$ colors are used. In contrast, we also exhibit infinite families of nonplanar bipartite cubic (and thus 3-edge colorable) graphs with a range of numbers of edge-Kempe equivalence classes when using 3 colors. These results address a question raised by Mohar.

1. Introduction and summary

Back in the frosts of time, Alfred Bray Kempe introduced the notion of changing colorings by switching maximal two-color chains of vertices (for vertex colorings) [4] or edges (for edge colorings). The maximal two-color chains are now called *Kempe chains* and *edge-Kempe chains* respectively; switching the colors along such a chain is called a *Kempe switch* or *edge-Kempe switch* as appropriate. This process is of interest across the study of colorings. It is also of interest in statistical mechanics, where certain dynamics in the antiferromagnetic q-state Potts model correspond to Kempe switches on vertex colorings [8,9]. In some cases, these dynamics also correspond to edge-Kempe switches [7].

In the present work we are concerned with understanding when two edge-colorings are equivalent under a sequence of edge-Kempe switches without introducing additional colors and when not. We will say that two n-colorings are edge Kempe-equivalent if they are equivalent without introducing additional colors. We allow multiple edges on our (labeled) graphs; loops are prohibited (and will mostly be excluded by other constraints such as 3-edge colorability).

A single edge-Kempe switch is denoted by—. That is, if coloring c_i becomes coloring c_j after a single edge-Kempe switch, then $c_i - c_j$. If coloring c_j can be converted to coloring c_k by a sequence of edge-Kempe switches, then c_j and c_k are equivalent; we denote this by $c_j \sim c_k$. Because $\sim$ is an equivalence relation, we may consider the equivalence classes on the set of colorings of a graph G edge-colored with n colors. In this paper we focus on the *number* of edge-Kempe equivalence classes and denote this quantity by $K'(G, n)$. (In other work this has been denoted $\mathrm{Ke}(L(G), n)$ [6] and $\kappa_E(G, n)$ [5].)

Note that any global permutation of colors can be achieved by edge-Kempe switches because the symmetric group S_n is generated by transpositions. Thus two colorings that differ only by a permutation of colors are edge-Kempe equivalent for any number of colors.

Recall that $\Delta(G)$ is the largest vertex degree in G and that $\chi'(G)$ is the smallest number of colors needed to properly edge-color G. Mohar has shown that when more colors are used than possibly needed to edge-color the graph, then

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http://dx.doi.org/10.1016/j.disc.2014.02.014

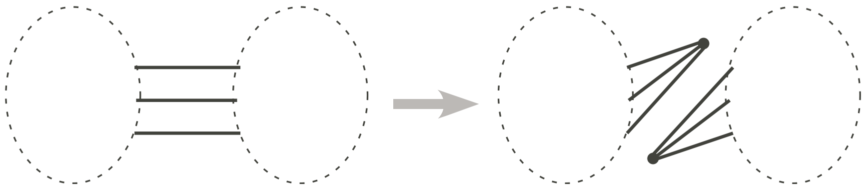

Fig. 1. Decomposing a graph over a 3-edge cut.

there is but a single edge-Kempe equivalence class, i.e., when $n > \chi'(G) + 1$ then $K'(G, n) = 1$ [6, Theorem 3.1]. More is known if $\Delta(G)$ is restricted; when $\Delta(G) \leq 4, K'(G, \Delta(G) + 2) = 1$ [5, Theorem 2] and when $\Delta(G) \leq 3, K'(G, \Delta(G) + 1) = 1$ [5, Theorem 3]. For bipartite graphs there is a stronger result: when $n > \Delta(G), K'(G, n) = 1$ [6, Theorem 3.3]. Little is known about $K'(G, \Delta(G))$ except that there are cases in which $K'(G, \Delta(G)) > 1$.

In this paper we focus on 3-edge colorable cubic graphs, and examine $K'(G, \chi'(G)) = K'(G, 3)$. Mohar suggested classifying cubic bipartite graphs with $K'(G, 3) = 1$ [6]; we provide a partial answer here. Mohar also points out in [6] that it follows from a result of Fisk in [2] that every planar 3-connected cubic bipartite graph G has $K'(G, 3) = 1$. We show (in Section 4) that for G planar, bipartite and cubic, G has $K'(G, 3) = 1$.

The remainder of the paper proceeds as follows. Section 2 introduces a decomposition of cubic graphs along 3-edge cuts that preserve planarity and bipartiteness. The theorems in Section 3 use this edge-cut decomposition to combine and decompose 3-edge colorings. We also show that any edge-Kempe equivalence can avoid color changes at a particular vertex. Then, in Section 4 we compute $K'(G, 3)$ in terms of the edge-cut decomposition of G, and exhibit infinite families of simple nonplanar bipartite cubic graphs with a range of numbers of edge-Kempe equivalence classes.

2. Decompositions of cubic graphs

Any 3-edge cut of a cubic graph may be used to decompose a cubic graph G into two cubic graphs G_1, G_2 as follows. For 3-edge cut $E_C = \{(s_{11}s_{21}), (s_{12}s_{22}), (s_{13}s_{23})\}$ where vertices s_{1j} are on one side of the cut and s_{2j} on the other, let the induced subgraphs of $G \setminus E_C$ separated by E_C be G'_1, G'_2. Then for $i = 1, 2$ define G_i by $V(G_i) = V(G'_i) \cup v_i$ and $E(G_i) = E(G'_i) \cup E_{C_i}$ where $E_{C_i} = \{(v_i s_{ij}) | j = 1, 2, 3\}$, as is shown in Fig. 1. Note that if $s_{ij} = s_{ik}$ then G_i will contain a multiple edge. This decomposition will be written as $G = G_1 \curlyvee G_2$.

We say the edge cut is nontrivial if both G_1 and G_2 have fewer vertices than G. Using nontrivial edge cuts, we may decompose a cubic graph G into a set of smaller graphs $\{G_i\}$ where each G_i has no nontrivial edge cuts (but may have additional multiple edges).

Notice that this decomposition is reversible, though not uniquely so. Consider two cubic graphs G_1, G_2. Form $G_1 \curlyvee G_2$ by distinguishing a vertex on each (v_1, v_2 respectively) and identifying the edges incident to v_1 with the edges incident to v_2. *A priori*, there are many ways to choose v_1, v_2 and many ways to identify their incident edges. We will abuse the notation $G_1 \curlyvee G_2$ by using it to denote a particular one of these many choices. It is easy to see that this operation preserves many properties. We highlight two useful properties here; the proofs are straightforward.

Lemma 2.1. *Let G be a cubic graph. If $G = G_1 \curlyvee G_2$ then G is planar if and only if G_1 and G_2 are planar.*

Recall that if G is cubic and class 1 it must be bridgeless. Note that if G_1 and G_2 are connected graphs that have bridges, then $G = G_1 \curlyvee G_2$ may have one, two, or three components depending on whether the cut edges are bridges and how they are identified.

Lemma 2.2. *Let G be a cubic graph. If $G = G_1 \curlyvee G_2$ then G is bipartite if and only if G_1 and G_2 are bipartite.*

Proof. Suppose G is a bipartite cubic graph with nontrivial 3-edge cut E_C and G'_1, G'_2 the induced subgraphs of $G \setminus E_C$. For a bipartition of G to descend naturally to bipartitions of G_1, G_2, the edges of E_C must be incident only to vertices in G'_i that are in the same part of G. Therefore, assume this is not the case and (without loss of generality) that two of the edges of E_C are incident to one part of G'_1 and the remaining edge of E_C is incident to the other part of G'_1. Let G'_1 have m_j vertices belonging to part j of G. There are $3m_1 - 1$ edges emanating from part 1 of G'_1 that must be incident to vertices of part 2 of G'_1. On the other hand, there are $3m_2 - 2$ edges emanating from part 2 of G'_1 that must be incident to vertices in part 1. Thus $3m_1 - 1 = 3m_2 - 2$, which is impossible.

Conversely, if G_1, G_2 are bipartite, with distinguished v_1, v_2 for the purpose of forming $G_1 \curlyvee G_2$, then use the bipartition of G_1 and assign v_2 to the opposite part as v_1 to induce a bipartition of $G_1 \curlyvee G_2$. □

Theorem 2.3. *A cubic graph H that is 2-connected but not 3-connected may be decomposed via $\curlyvee$ into a set of cubic loopless graphs $\{H_i\}$ where each H_i is 3-connected.*

Proof. The proof is inductive on the number of vertices of H. Because H is 2-connected but not 3-connected, there exists a 2-vertex separating set. Fig. 2 shows the three possible edge configurations for a 2-vertex separating set of a cubic graph,