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Note

A note on graphs contraction-critical with respect to independence number

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ABSTRACT

Let $\alpha(G)$ denote the independence number of graph G, i.e., the size of any maximum independent set of vertices. A graph G is *contraction-critical* (with respect to α) if $\alpha(G/\widehat{xy}) < \alpha(G)$, for every edge $xy \in E(G)$, where $G/\widehat{xy}$ denotes the graph obtained from G by shrinking the edge xy to a single vertex $\widehat{xy}$ and deleting the loop thus formed. Let us denote the class of all such contraction-critical graphs by CCR.

First it is shown that all graphs in CCR must be bipartite.

Let $G = (A, B)$ be a bipartite graph with bipartition $A \cup B$. It is then shown that (a) if $|A| < |B|$, then $G \in CCR$ if and only if B is the unique maximum independent set in G if and only if G is 1-expanding, while (b) if $|A| = |B|$, then $G \in CCR$ if and only if A and B are the only two maximum independent sets in G if and only if G is 1-extendable.

It follows that CCR graphs can be recognized in polynomial time.

1. Introduction

Let $\alpha(G)$ denote the independence number of the graph G, that is, the size of any largest independent set of vertices in G. Problems involving $\alpha(G)$ have proven to be quite difficult on the whole so far. Perhaps this is not so surprising in view of the fact that the determination of $\alpha(G)$ for arbitrary G was one of the first problems to be shown to be NP-complete [10]. On the other hand, for certain special subclasses of graphs, $\alpha(G)$ can be computed in polynomial time. For example, if G is bipartite, then $\alpha(G)$ may be computed in polynomial time by using a polynomial matching algorithm together with two classical results by König and Gallai (cf. [15]). Claw-free graphs constitute a second graph class admitting a polynomial algorithm for determining α (cf. [16,20]).

Clearly, if a graph has the property that every maximal independent set is, in fact, maximum (the so-called *well-covered* graphs), determination of $\alpha(G)$ becomes trivially polynomial, but no polynomial recognition algorithm for well-covered graphs is known. Interesting results about well-covered graphs have been obtained, however. For a sampling, see the surveys [5,17].

Historically, useful information about certain graph parameters has been gleaned from the study of various kinds of "criticality" with respect to the given parameter. For example, graphs which are critical with respect to edge *deletion* (the so-called α-*critical* graphs) have been investigated. These are the graphs G for which $\alpha(G - e) > \alpha(G)$, for every edge $e \in E(G)$. However, no polynomial recognition algorithm for this graph family is known and these graphs remain far from well-understood. For more on this graph family, see Section 12.1 of [15].

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The idea of criticality with respect to edge-*contraction* has proved extremely useful in the study of certain graph properties, notably in the study of graph connectivity (cf. [12]), graph coloring (cf. [9,13,21]) and graph minors (cf. [19]). However, the notion of criticality of edge contraction applied to independence number does not seem to have been investigated heretofore.

Let G be a graph and xy an edge in G. The edge xy is be said to be *contraction-critical* (with respect to α) if $\alpha(G/\widehat{xy}) < \alpha(G)$, where $G/\widehat{xy}$ denotes the graph obtained from G by shrinking the edge xy to a single vertex $\widehat{xy}$ and deleting the loop and any multiple edges thus formed. A graph G is *contraction-critical* if every edge of G is contraction-critical. Let us denote the class of all such graphs by *CCR*. (Note that a graph belongs to *CCR* if and only if all its components belong to *CCR*.)

In the present note, we show, that the *CCR* graphs turn out to be a well-known polynomially recognizable family in disguise. First we show that all members of the family *CCR* must be bipartite and then we obtain a characterization theorem. To accomplish this, we shall need the notions of 1-*extendability* and 1-*expandability* both of which deal with matchings. Although these two properties may be defined for graphs in general, we will confine both properties to bipartite graphs since the class *CCR* is bipartite. So let $G = (A, B)$ be a bipartite graph with bipartition $A \cup B$ where, without loss of generality, we assume that $|A| \le |B|$.

Definition (*[1]*)**.** If $G = (A, B)$ is bipartite with $|A| < |B|$, then G is 1-*expanding* if for every edge e of G, there is a matching of all A into B which contains e.

An equivalent version of this property is given by the following result.

Lemma 1.1 (*Lemma 6.2.4 [1]*)**.** *A connected bipartite graph $G = (A, B)$ with $|A| < |B|$ is 1-expanding if and only if for all $X \subseteq A$, $\emptyset \ne X \ne A$, $|N(X)| \ge |X| + 1$.*

The "balanced" version of the above property is the following.

Definition (*[6,15]*)**.** If $G = (A, B)$ is bipartite with $|A| = |B|$, then G is 1-*extendable* if every edge of G lies in a perfect matching in G.

This property too admits a useful equivalent statement.

Lemma 1.2 (*[6,14]*)**.** *A connected bipartite graph $G = (A, B)$ with $|A| = |B|$ is 1-extendable if and only if for all $X \subseteq A$, $\emptyset \ne X \ne A$, $|N(X)| \ge |X| + 1$.*

In this note we first show that if $G \in CCR$, it must be bipartite. We then obtain the following characterizations: (a) if $G = (A, B)$ is a connected bipartite graph with $|A| < |B|$, then $G \in CCR$ if and only if B is the unique maximum independent set in G if and only if G is 1-expanding, while (b) if $G = (A, B)$ is a connected bipartite graph with $|A| = |B|$, then $G \in CCR$ if and only if A and B are the only two maximum independent sets in G if and only if G is 1-extendable.

It follows from these characterizations that, since both the 1-extendable and 1-expanding properties can be checked in polynomial time via matching algorithms, membership in the class *CCR* is also polynomially verifiable.

We shall denote by $\nu(G)$ the size of a maximum matching in graph G and by $\tau(G)$ the size of a minimum vertex cover in G. Note that we will not consider the single-vertex graph K_1 to be bipartite. For all other terminology and notation, see [2]. Note also that in this paper all graphs are finite and simple.

2. Main results

Lemma 2.1. *If G belongs to CCR, then G is a bipartite graph and every maximum independent set is a partite set of G. In particular, $|V(G)| \le 2\alpha(G)$.*

Proof. Let G be a contraction-critical graph and let I be a maximum independent set in G. If $G - I$ contains an edge e, let $G' = G/\hat{e}$. Then I is an independent set in G' which implies that $\alpha(G') \ge \alpha(G)$. This contradicts the definition of a contraction-critical graph. Therefore, $V(G) - I$ is an independent set. Since I is also independent, G is a bipartite graph with bipartition $(I, V(G) - I)$.

Since I is a maximum independent set and $V(G) - I$ is an independent set, $|V(G) - I| \le |I| = \alpha(G)$ and hence $|V(G)| \le 2\alpha(G)$. ■

The following result is well-known.

Lemma 2.2. *If G is a connected bipartite graph with bipartition (A, B), then this bipartition is unique.* ■

Corollary 2.3. *Suppose $G \in CCR$ with bipartition (A, B), where by Lemma 2.1 we may assume $|B| = \alpha(G)$. Then if $|A| < |B|$, G contains exactly one maximum independent set, namely B, and if $|A| = |B|$, G contains exactly two maximum independent sets, namely A and B.*