Discrete Mathematics 320 (2014) 1–14

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Graphs with equal chromatic symmetric functions

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ARTICLE INFO

Article history:
Received 5 September 2013
Received in revised form 1 December 2013
Accepted 7 December 2013
Available online 17 December 2013

Keywords:
Chromatic symmetric function
Graph coloring
Unicyclic graphs
Trees

ABSTRACT

In 1995 Stanley introduced the chromatic symmetric function $\mathbf{X}_G$ associated to a simple graph G as a generalization of the chromatic polynomial of G. In this paper we present a novel technique to write $\mathbf{X}_G$ as a linear combination of chromatic symmetric functions of smaller graphs. We use this technique to give a sufficient condition for two graphs to have the same chromatic symmetric function. We then construct an infinite family of pairs of unicyclic graphs with the same chromatic symmetric function, answering the question posed by Martin, Morin, and Wagner of whether such a pair exists. Finally, we approach the problem of whether it is possible to determine a tree from its chromatic symmetric function. Working towards an answer to this question, we give a classification theorem for single-centroid trees in terms of data closely related to its chromatic symmetric function.

0. Introduction

In 1995, Stanley [12] introduced a symmetric function $\mathbf{X}_G = \mathbf{X}_G(x_1, x_2, \ldots)$ associated with any simple graph G (see Section 1 for a precise definition) called the *chromatic symmetric function* of G. $\mathbf{X}_G$ has the property that when we specialize the variables to $x_1 = \cdots = x_k = 1$ and $x_i = 0$ for all $i > k$ then $\mathbf{X}_G$ gives the number of ways to properly color the vertices of G with k colors. Hence $\mathbf{X}_G(1, 1, \ldots, 1, 0, \ldots) = \chi_G(k)$, where χ_G is the chromatic polynomial of G.

One of the first questions posed by Stanley was whether $\mathbf{X}_G$ determines G. As expected this is not the case, and Stanley provides the example of the kite and the bowtie as nonisomorphic graphs with the same $\mathbf{X}_G$ [12, Fig. 1]. Although two nonisomorphic graphs may share the same chromatic symmetric function, Stanley conjectured that two nonisomorphic trees must have distinct chromatic symmetric functions. This conjecture is true for trees with 25 or fewer vertices as computed by Keeler Russell in [10]. Other evidence that Stanley's conjecture is true has been found by Morin [9] and Fougere [4] who showed that some families of trees are determined by the chromatic symmetric function. Martin, Morin and Wagner [8] showed that the degree sequence and path sequence of a tree, T, can be obtained from $\mathbf{X}_T$. They also showed that some families of trees, called caterpillars and spiders, can be determined from their chromatic symmetric function.

A fundamental property of the chromatic polynomial is the *deletion–contraction* property, which allows us to write $\chi_G(k)$ as a linear combination of the chromatic polynomials of graphs with fewer edges. This property is the basis for inductive proofs of many other properties of the chromatic polynomial. Unfortunately $\mathbf{X}_G$ does not satisfy a deletion–contraction law which makes it difficult to apply the useful technique of induction. Gebhard and Sagan [5] introduced a non-commutative version of $\mathbf{X}_G$ that satisfies the deletion–contraction property and is a complete invariant of graphs. One of our results is a technique to decompose $\mathbf{X}_G$ as a linear combination of chromatic symmetric functions of other graphs. And in the case when G has a triangle we can write $\mathbf{X}_G$ as a linear combination of chromatic symmetric functions of graphs with fewer edges than G.

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http://dx.doi.org/10.1016/j.disc.2013.12.006

There are many properties of G that can be recovered from $\mathbf{X}_G$. These include the number of vertices, the number of connected components, the number of matchings, and the girth. We have found that the number of triangles in G can also be recovered from $\mathbf{X}_G$. In the case that the graph is a tree, T, a lot more can be recovered from $\mathbf{X}_T$; for example, the degree sequence can be recovered from $\mathbf{X}_T$ [8]. This is no longer true for general graphs, the bowtie/kite pair of graphs were shown by Stanley to have the same chromatic symmetric function but different degree sequences. Although the degree sequence can no longer be recovered from $\mathbf{X}_G$ for arbitrary G, we show that the sum of the squares of the degrees can be recovered from $\mathbf{X}_G$. This is a generalization of a result in [4] that shows the analogous result for trees.

In [8] the authors showed that $\mathbf{X}_G$ is an complete invariant for two special families of unicyclic graphs and asked whether there exists a pair of unicyclic graphs with the same $\mathbf{X}_G$. We answer this question in the affirmative by giving a pair of unicyclic graphs with the same chromatic symmetric function. In fact, our Theorem 4.2 gives a sufficient condition for two graphs to have the same chromatic symmetric function. We apply this theorem to construct infinitely many pairs of unicyclic graphs with the same $\mathbf{X}_G$. The same technique can also be used to construct pairs of general graphs with the same $\mathbf{X}_G$. We have also studied trees and we give a classification theorem for trees with one centroid. This classification arose from our study of the chromatic symmetric function of a tree when written in the power-sum symmetric basis.

Our paper is organized as follows. In Section 1 we review background information, set up notation, and define the chromatic symmetric function. In Section 2 we look at properties of G that are determined by $\mathbf{X}_G$ for general graphs. In particular, we show that the sum of the squares of the degrees as well as the number of triangles in a graph can be recovered from the chromatic symmetric function. In Section 3 we show how the chromatic symmetric function of a graph can be written as a linear combination of other chromatic symmetric functions. In Section 4 we focus our attention on unicyclic graphs. We also prove a sufficient condition for two graphs to have the same chromatic symmetric function and show how to construct pairs of graphs with the same $\mathbf{X}_G$. In our last section, Section 5, we prove a classification theorem for trees with a single centroid that is closely related to the coefficients of the chromatic symmetric function when written in the power-sum symmetric basis.

1. Preliminaries

We assume that the reader is familiar with the basic facts about graphs found in any introductory graph theory book (see e.g., [1,3,7]). In this section we establish notation that will be used throughout the paper. A graph G is an ordered pair (V, E), where $V = V(G)$ is the vertex set and $E = E(G)$ is the edge set. All our graphs are *simple*, i.e., we do not allow loops or multiple edges. The number of vertices $\#V(G)$ is called the *order* of the graph. We will write uv for the edge joining the vertices $u, v \in V(G)$ if such an edge exists. We say that u and v are *endpoints* of uv, that uv is *incident* to u and v, and that u is *adjacent* to v. If two edges have no endpoints in common, they are *disjoint*. The *degree* $d(v)$ of a vertex v is the number of edges incident to v. The *degree sequence* of a graph G is the non-increasing sequence of its vertex degrees. An *isolated* vertex is a vertex of degree 0. A *leaf* is a vertex of degree 1. The *girth* of a graph is the number of distinct vertices in a shortest cycle in the graph. An acyclic graph has infinite girth.

A *subgraph* $G' \subseteq G$ of a graph $G = (V(G), E(G))$ is a graph $G' = (V'(G), E'(G))$ such that $V'(G) \subseteq V(G)$ and $E'(G) \subseteq E(G)$. A subgraph is said to be *induced* by the vertex set $V'(G)$ if every edge in $E(G)$ having endpoints in $V'(G)$ is also in $E'(G)$. A subgraph H is a *spanning subgraph* of G if it has the same vertex set as G. A subgraph is said to be a *matching* of size k if it consists of k disjoint edges on $2k$ vertices.

In this paper we are interested in certain classes of simple graphs. A graph is called *unicyclic* if it contains exactly one cycle, a *forest* if it contains no cycles, and a *tree* if it is a connected forest. Notice that a connected unicyclic graph with n vertices has n edges.

In the following proposition we summarize some well-known facts about trees. The reader may refer to [1,3,7] or any other introductory graph theory textbook for proofs of these facts.

Proposition 1.1 (*[1, pp. 99–100]*)**.**

(1) *In a tree, any two vertices are connected by exactly one path.*
(2) *Every tree on n vertices has $n - 1$ edges. In general, a forest on n vertices with c connected components has $n - c$ edges.*
(3) *Every nontrivial tree has at least two leaves. In general, if a forest contains c connected nontrivial components, then it contains at least $2c$ leaves.*

We now give two definitions that are not as standard as the others we have given so far. We will use these definitions in Section 5. For further reading on these concepts see [7].

Definition 1.2. The *weight* of a vertex v of a tree T is the maximal number of edges in any subtree of T containing v as a leaf.

Definition 1.3. The *centroid* of a tree T is the set of all vertices of T having minimum weight.

An example of the weights of vertices of a tree is shown in Fig. 1. In that graph, the vertex with weight 8 is the centroid of the tree.

Proposition 1.4 (*[1, p. 99]*)**.** *Every tree has a centroid consisting of either one vertex or two adjacent vertices (see Fig. 2).*