Discrete Mathematics 315–316 (2014) 9–17

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

A weight statistic and partial order on products of m-cycles

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ARTICLE INFO

Article history:
Received 20 March 2013
Received in revised form 26 September 2013
Accepted 2 October 2013
Available online 18 October 2013

Keywords:
Symmetric group
m-cycles
Bruhat order
EL-shellable posets

ABSTRACT

R. S. Deodhar and M. K. Srinivasan defined a weight statistic on the set of involutions in the symmetric group and proved several results about the properties of this weight. These results include a recursion for a weight generating function, that the weight provides a grading for the set of fixed-point free involutions under a partial order related to the Bruhat partial order, and that this graded poset is EL-shellable and its order complex triangulates a ball. We extend the definition of weight to products of disjoint m-cycles in the symmetric group, and we generalize all of the results of Deodhar and Srinivasan just mentioned to the case of any $m \geq 2$.

1. Introduction

Let m, n be integers, with $2 \leq m \leq n$, let S_n be the symmetric group on $[n] = \{1, 2, \ldots, n\}$, and let $\delta \in S_n$ be a product of disjoint m-cycles. In particular, if m is prime, then δ is just an element of order m in S_n (or the identity, if it is the product of zero m-cycles). Writing δ in cycle notation, suppose we have δ is a product of k disjoint m-cycles (so $mk \leq n$), so that

$$\delta = (a_{1,1}\ a_{1,2} \cdots\ a_{1,m})(a_{2,1}\ a_{2,2}\ \cdots\ a_{2,m}) \cdots (a_{k,1}\ a_{k,2}\ \cdots\ a_{k,m}). \tag{1.1}$$

Further, suppose that for each $i = 1, 2, \ldots, k$, $a_{i,1} < a_{i,j}$ for $j = 2, 3, \ldots, m$, and $a_{1,1} < a_{2,1} < \cdots < a_{k,1}$, and we then say that δ is in *standard form*. Let $J^{(m)}(n)$ denote the collection of all products of disjoint m-cycles in S_n, and let $J^{(m)}(n, k)$ denote the collection of all products of k disjoint m-cycles in S_n, so that $\delta \in J^{(m)}(n, k)$ in (1.1). Given $\delta \in J^{(m)}(n, k)$ in standard form as in (1.1), define span(δ) as

$$\mathrm{span}(\delta) = \sum_{i=1}^{k} \sum_{j=2}^{m} (a_{i,j} - a_{i,1} - 1).$$

For example, suppose $\delta \in J^{(3)}(9, 3)$, where $\delta = (1\ 6\ 9)(2\ 7\ 4)(3\ 5\ 8)$. Then

$$\mathrm{span}(\delta) = (5 - 1) + (8 - 1) + (5 - 1) + (2 - 1) + (2 - 1) + (5 - 1) = 21.$$

Given an m-cycle $(a_1\ a_2 \ldots a_m) \in S_n$ in standard form, draw its *arc diagram* by drawing, along a line containing points labeled from $[n]$, an arc for each pair (a_1, a_j), $j = 2, \ldots, m$, where the arc (a_1, a_l) is drawn under (a_1, a_j) when $l > j$. When $a_j > a_l$ and $j > l$, then the arcs (a_1, a_j) and (a_1, a_l) intersect in the arc diagram, which we call an *internal crossing* of the m-cycle. That is, the number of internal crossings of the m-cycle $(a_1\ a_2 \ldots a_m)$ is equal to the number of pairs (a_i, a_j) from the sequence $a_2, a_3, \ldots, a_m$, which satisfy $i < j$ and $a_i < a_j$, which we call *ascents* of this sequence (which are also known as non-inversions). In Figs. 1 and 2, for example, we show the arc diagrams for the 5-cycles (1 4 5 3 2) and (1 2 4 5 3), with the internal crossings circled.

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0012-365X/$ – see front matter
http://dx.doi.org/10.1016/j.disc.2013.10.002

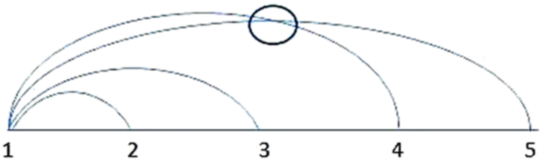

Fig. 1. Arc diagram and internal crossing of (1 4 5 3 2).

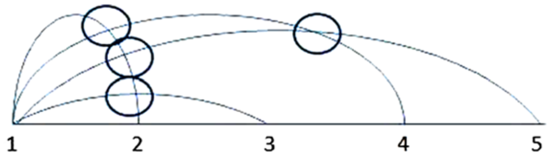

Fig. 2. Arc diagram and internal crossings of (1 2 4 5 3).

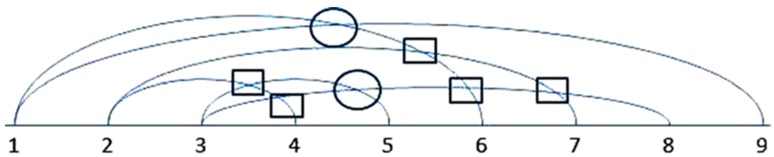

Fig. 3. Arc diagram and all crossings of (1 6 9) (2 7 4) (3 5 8).

If $\delta \in J^{(m)}(n, k)$, the arc diagram for δ is drawn by drawing the arc diagram for each of the k disjoint m-cycles of δ. There may be intersections of the arcs from different m-cycles of δ, which we call *external crossings* of δ. Let $C_{\text{in}}(\delta)$ denote the total number of internal crossings in the arc diagram of δ, and $C_{\text{ex}}(\delta)$ denote the total number of external crossings in the arc diagram of δ. The *crossing number* of δ, $C(\delta)$, is then defined as $C(\delta) = C_{\text{ex}}(\delta) + C_{\text{in}}(\delta)$. In Fig. 3, we show the arc diagram for $\delta = (1\ 6\ 9)(2\ 7\ 4)(3\ 5\ 8) \in J^{(3)}(9, 3)$, with internal crossings in circles and external crossings in boxes.

Now, given any $\delta \in J^{(m)}(n)$, we define the *weight* of δ, which we denote $\text{wt}_m(\delta)$, as

$$\text{wt}_m(\delta) = \text{span}(\delta) - C(\delta) = \text{span}(\delta) - C_{\text{ex}}(\delta) - C_{\text{in}}(\delta).$$

So, for $\delta = (1\ 6\ 9)(2\ 7\ 4)(3\ 5\ 8)$, since $\text{span}(\delta) = 21$ and $C(\delta) = 7$ from Fig. 3, then $\text{wt}_3(\delta) = 14$. We note that for $m = 2$, our definition of weight coincides precisely with that from [5], since if $\delta \in S_n$ is an involution, then $C_{\text{in}}(\delta) = 0$.

Given integers $n, k \geq 0$, $m \geq 2$, with $mk \leq n$, define the *weight generating function*, denoted $j_q^{(m)}(n, k)$, to be the following polynomial in an indeterminate q:

$$j_q^{(m)}(n, k) = \sum_{\delta \in J^{(m)}(n,k)} q^{\text{wt}_m(\delta)}.$$

In Section 2, we prove a recursive relation on the weight generating function in terms of n and k (Theorem 2.1), and we use it to compute an exact formula (Corollary 2.1) for $j_q^{(m)}(mn, n)$, the weight generating function for the set of products of disjoint m-cycles in S_{mn} which are fixed-point free. We use the notation $F^{(m)}(mn) = J^{(m)}(mn, n)$ for the set of fixed-point free products of disjoint m-cycles in S_{mn}.

In Section 3, we introduce the Bruhat order (sometimes called the *strong* Bruhat order) on the symmetric group S_n, which makes S_n a graded poset with grading given by the number of inversions of a permutation. We consider a specific subset $E(n_m)$ of a permutation group $S(n_m)$ (identified with S_{mn}), and the set $E(n_m)$ is in bijection with the set $F^{(m)}(mn)$ of fixed-point free products of disjoint m-cycles in S_{mn}. In Proposition 3.1, we show that an explicit bijection ϕ defined between these two sets maps the weight wt_m of a permutation in $F^{(m)}(mn)$ to the number of inversions of the permutation in $E(n_m)$.

Now let $\delta, \pi \in F^{(m)}(mn)$, and suppose that

$$\delta = (a_{1,1}\ a_{1,2} \cdots\ a_{1,m})(a_{2,1}\ a_{2,2}\ \cdots\ a_{2,m}) \cdots (a_{n,1}\ a_{n,2}\ \cdots\ a_{n,m})$$

is in standard form. Then π is obtained from δ by an *interchange* if one of the following holds:

(i) There is some i, $1 \leq i \leq n$, and some j, l, $2 \leq j, l \leq m$, such that the standard form of π is obtained by interchanging $a_{i,j}$ and $a_{i,l}$ in δ.
(ii) There are some i, j, $1 \leq i < j \leq n$, and some l, $2 \leq l \leq m$, such that the standard form of π is obtained by interchanging $a_{i,l}$ and $a_{j,1}$ in δ.
(iii) There are some i, j, l, h, $2 \leq l, h \leq m$, $l \neq h$, $1 \leq i < j \leq n$, such that the standard form of π is obtained by interchanging $a_{i,l}$ and $a_{j,h}$ in δ.