Discrete Mathematics 315–316 (2014) 18–28

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

A new characterization of taut distance-regular graphs of odd diameter

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ARTICLE INFO

Article history:
Received 31 May 2013
Received in revised form 1 October 2013
Accepted 5 October 2013
Available online 26 October 2013

Keywords:
Distance-regular graph
Association scheme
Bipartite graph
Taut graph

ABSTRACT

We consider a bipartite distance-regular graph Γ with vertex set X, diameter $D \geq 4$, and valency $k \geq 3$. Let $\mathbb{C}^X$ denote the vector space over $\mathbb{C}$ consisting of column vectors with rows indexed by X and entries in $\mathbb{C}$. For $z \in X$, let $\hat{z}$ denote the vector in $\mathbb{C}^X$ with a 1 in the z^{th} row and 0 in all other rows. For $0 \leq i \leq D$, let $\Gamma_i(z)$ denote the set of vertices in X that are distance i from z. Fix $x, y \in X$ with distance $\partial(x, y) = 2$. For $0 \leq i, j \leq D$, we define $w_{ij} = \sum \hat{z}$, where the sum is over all vertices $z \in \Gamma_i(x) \cap \Gamma_j(y)$. Define a parameter Δ in terms of the intersection numbers by $\Delta = (b_1 - 1)(c_3 - 1) - (c_2 - 1)p^2_{22}$. For $2 \leq i \leq D - 2$ we define vectors $w^+_{ii} = \sum |\Gamma_1(x) \cap \Gamma_1(y) \cap \Gamma_{i-1}(z)|\hat{z}$, where the sum is over all vertices $z \in \Gamma_i(x) \cap \Gamma_i(y)$. We define $W = \text{span}\{w_{ij}, w^+_{hh} \mid 0 \leq i, j \leq D,\ 2 \leq h \leq D - 2\}$. In [M. MacLean, An inequality involving two eigenvalues of a bipartite distance-regular graph, Discrete Math. 225 (2000) 193–216], MacLean defined what it means for Γ to be *taut*. Assume D is odd. We show Γ is taut if and only if $\Delta \neq 0$ and the subspace W is invariant under multiplication by the adjacency matrix.

1. Introduction

Throughout this introduction, let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$, valency $k \geq 3$, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_D$ (see Section 2 for formal definitions). In [14, Theorem 5.11], we showed that the intersection numbers of Γ satisfy

$$b_3\big((k-2)b_2 - \theta_1^2(\mu - 1)\big)\big((k-2)b_2 - \theta_d^2(\mu - 1)\big) \geq b_1\Delta(\theta_1^2 - b_2)(b_2 - \theta_d^2), \tag{1}$$

where $d = \lfloor D/2 \rfloor$, and where $\Delta = (b_1 - 1)(c_3 - 1) - (\mu - 1)p^2_{22}$. When $\Delta = 0$, equality holds in (1) precisely when Γ is 2-homogeneous in the sense of Curtin [4] and Nomura [18]. We defined Γ to be *taut* whenever $\Delta \neq 0$ and equality holds in (1). In this paper, we obtain a new characterization of the taut condition in the case where D is odd.

Let X denote the vertex set of Γ, and let $\mathbb{C}^X$ denote the vector space over $\mathbb{C}$ consisting of column vectors with rows indexed by X and entries in $\mathbb{C}$. For $z \in X$, let $\hat{z}$ denote the vector in $\mathbb{C}^X$ with a 1 in the zth row and 0 in all other rows. For $0 \leq i \leq D$, let $\Gamma_i(z)$ denote the set of vertices in X that are distance i from z. Fix $x, y \in X$ with distance $\partial(x, y) = 2$. For $0 \leq i, j \leq D$, we define $w_{ij} = \sum \hat{z}$, where the sum is over all vertices $z \in \Gamma_i(x) \cap \Gamma_j(y)$. For $2 \leq i \leq D-1$ we define vectors $w^+_{ii} = \sum |\Gamma_1(x) \cap \Gamma_1(y) \cap \Gamma_{i-1}(z)|\hat{z}$, where the sum is over all vertices $z \in \Gamma_i(x) \cap \Gamma_i(y)$.

To provide context for our work, we now mention two theorems proven by Curtin and Miklavic, respectively. For notational purposes, let $\mathcal{L} = \{ij \mid 0 \leq i, j \leq D,\ i + j \geq 2,\ |i - j| \in \{0, 2\}\}$.

Theorem 1.1 ([4, Theorem 17]). *The following are equivalent.*

(i) *Γ is 2-homogeneous.*
(ii) *The vector space* $\text{span}\{w_{ij} \mid ij \in \mathcal{L}\}$ *is invariant under multiplication by the adjacency matrix.*

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0012-365X/$ – see front matter
http://dx.doi.org/10.1016/j.disc.2013.10.005

Theorem 1.2 (*[17, Theorems 11.4, 11.6]*)**.** *Assume Γ is Q-polynomial (see [2] for the definition). Then* (i)–(ii) *hold below.*

(i) *Assume Γ is the antipodal quotient of the 2D-cube. Then the vector space* $\text{span}\{w_{ij}, w^+_{D-1,D-1} \mid ij \in \mathcal{L}\}$ *is invariant under multiplication by the adjacency matrix.*

(ii) *Assume Γ is neither 2-homogeneous nor the antipodal quotient of the 2D-cube. Then the vector space* $\text{span}\{w_{ij}, w^+_{hh} \mid ij \in \mathcal{L},\ 2 \le h \le D-1\}$ *is invariant under multiplication by the adjacency matrix.*

In this paper, we prove the following results. The first provides a new characterization of taut distance-regular graphs of odd diameter.

Theorem 1.3. *Assume D is odd. Then the following are equivalent.*

(i) *Γ is taut.*

(ii) *$\Delta \neq 0$ and the vectors $\{w_{ii}, w_{i-1,i+1}, w_{i+1,i-1} \mid 1 \le i \le D-1\} \cup \{w^+_{ii} \mid 2 \le i \le D-2\}$ form a basis for a subspace that is invariant under multiplication by the adjacency matrix.*

To illustrate the meaning of Theorem 1.3, we prove the following result, which presents a combinatorial implication of Γ being taut.

Theorem 1.4. *Assume Γ is taut with odd diameter. Then there exist real scalars η_i, ψ_i $(2 \le i \le D-2)$ such that for all $z \in \Gamma_i(x) \cap \Gamma_i(y)$,*

$$|\Gamma_{i-1}(x) \cap \Gamma_{i-1}(y) \cap \Gamma_1(z)| = \eta_i + \psi_i\, |\Gamma_1(x) \cap \Gamma_1(y) \cap \Gamma_{i-1}(z)| \quad (2 \le i \le D-2).$$

Finally, we remark that this paper is part of a continuing effort to understand and classify the bipartite distance-regular graphs with at most two irreducible T-modules of endpoint two, both of which are thin (see Section 7 for formal definitions). Please see [4,14,12,13,15,16] for more work from this ongoing project. We remark that in [15], Terwilliger and the current author outlined a number of problems for further research. Theorem 1.3 above provides a solution to the first part of Problem 17.12 in [15].

2. Preliminaries

In this section, we review some basic definitions and results. For more information, the reader may consult the books of Bannai and Ito [1], Brouwer, Cohen, and Neumaier [2], and Godsil [10].

Let X denote a nonempty finite set. Let $\text{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of all matrices whose rows and columns are indexed by X and whose entries are in $\mathbb{C}$. Let $V = \mathbb{C}^X$ denote the vector space over $\mathbb{C}$ consisting of column vectors whose coordinates are indexed by X and whose entries are in $\mathbb{C}$. We observe $\text{Mat}_X(\mathbb{C})$ acts on V by left multiplication. We endow V with the Hermitian inner product $\langle , \rangle$ which satisfies $\langle u, v\rangle = u^t\overline{v}$ for all $u, v \in V$, where t denotes transpose and $-$ denotes complex conjugation. For all $y \in X$, let $\hat{y}$ denote the element of V with a 1 in the y coordinate and 0 in all other coordinates. We observe $\{\hat{y} \mid y \in X\}$ is an orthonormal basis for V. We remark that for all $B \in \text{Mat}_X(\mathbb{C})$ and for all $u, v \in V$,

$$\langle Bu, v\rangle = \langle u, \overline{B}^t v\rangle. \tag{2}$$

Throughout this paper, let $\Gamma = (X, R)$ denote a finite, undirected, connected graph without loops or multiple edges, with vertex set X and edge set R. Let ∂ denote the path-length distance function for Γ, and set $D := \max\{\partial(x, y) \mid x, y \in X\}$. We refer to D as the *diameter* of Γ. For all $x \in X$ and for all integers i, we set $\Gamma_i(x) := \{y \in X \mid \partial(x, y) = i\}$.

The graph Γ is said to be *distance-regular* whenever for all integers h, i, j $(0 \le h, i, j \le D)$, and for all $x, y \in X$ with $\partial(x, y) = h$, the number

$$p^h_{ij} = |\Gamma_i(x) \cap \Gamma_j(y)|$$

is independent of the choice of x and y. The numbers p^h_{ij} are called *intersection numbers* of Γ. It is conventional to abbreviate $c_i = p^i_{1i-1}$ $(1 \le i \le D)$, $a_i = p^i_{1i}$ $(0 \le i \le D)$, $b_i = p^i_{1i+1}$ $(0 \le i \le D-1)$, and to define $c_0 = 0,\ b_D = 0$. We note $a_0 = 0$ and $c_1 = 1$. We abbreviate $\mu = c_2$.

For the rest of this paper we assume Γ is distance-regular with diameter $D \ge 3$. We observe Γ is regular with valency $k = b_0$ and that $c_i + a_i + b_i = k$ $(0 \le i \le D)$. For $0 \le i \le D$ we abbreviate $k_i = p^0_{ii}$, and observe

$$k_i = |\{z \in X \mid \partial(x, z) = i\}|, \tag{3}$$

where x is any vertex in X. Apparently $k_0 = 1$ and $k_1 = k$.

For each integer i $(0 \le i \le D)$, let A_i denote the matrix in $\text{Mat}_X(\mathbb{C})$ with x, y entry

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x, y) = i, \\ 0 & \text{if } \partial(x, y) \neq i \end{cases} \quad (x, y \in X). \tag{4}$$

For $0 \le i \le D$, we refer to A_i as the ith *distance matrix* of Γ. We abbreviate $A := A_1$ and note this is the adjacency matrix of Γ. Let M denote the subalgebra of $\text{Mat}_X(\mathbb{C})$ generated by A; M is known as the *Bose–Mesner algebra* of Γ. One can show that $A_0, A_1, \ldots, A_D$ form a basis for M [2, p. 44].