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Some enumerative results related to ascent sequences

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ABSTRACT

An *ascent sequence* is a sequence consisting of non-negative integers in which the size of each letter is restricted by the number of ascents preceding it in the sequence. Ascent sequences have recently been shown to be related to $(2 + 2)$-free posets and a variety of other combinatorial structures. In this paper, we prove some recent conjectures of Duncan and Steingrímsson concerning pattern avoidance for ascent sequences. Given a pattern τ, let $\mathcal{S}_\tau(n)$ denote the set of ascent sequences of length n avoiding τ. Here, we show that the joint distribution of the statistic pair (asc, zeros) on $\mathcal{S}_{0012}(n)$ is the same as (asc, RLmax) on the set of 132-avoiding permutations of length n. In particular, the ascent statistic on $\mathcal{S}_{0012}(n)$ has the Narayana distribution. We also enumerate $\mathcal{S}_\tau(n)$ when $\tau = 1012$ or $\tau = 0123$ and confirm the conjectured formulas in these cases. We combine combinatorial and algebraic techniques to prove our results, in two cases, making use of the kernel method. Finally, we discuss the case of avoiding 210 and determine two related recurrences.

1. Introduction

An *ascent* in a sequence $x_1x_2\cdots x_k$ is a place $j \geq 1$ such that $x_j < x_{j+1}$. An *ascent sequence* $x_1x_2\cdots x_n$ is one consisting of non-negative integers satisfying $x_1 = 0$ and for all i with $1 < i \leq n$,

$$x_i \leq \text{asc}(x_1x_2\cdots x_{i-1}) + 1,$$

where $\text{asc}(x_1x_2\cdots x_k)$ is the number of ascents in the sequence $x_1x_2\cdots x_k$. An example of such a sequence is 01013212524, whereas 01003221 is not, because 3 exceeds $\text{asc}(0100) + 1 = 2$. Starting with the paper by Bousquet-Mélou, Claesson, Dukes, and Kitaev [2], where they were related to the $(2 + 2)$-free posets and the generating function was determined, ascent sequences have since been studied in a series of papers where connections to many other combinatorial structures have been made. See, for example, [5,6,10] as well as [9, Section 3.2.2] for further information.

In this paper, we prove some conjectures made by Duncan and Steingrímsson [7] concerning the avoidance of patterns by ascent sequences. The patterns considered are analogous to patterns considered originally on permutations and later on other structures such as k-ary words and finite set partitions.

By a *pattern*, we will mean a sequence of non-negative integers, where repetitions are allowed. Let $\pi = \pi_1\pi_2\cdots\pi_n$ be an ascent sequence and $\tau = \tau_1\tau_2\cdots\tau_m$ be a pattern. We will say that π *contains* τ if π has a subsequence that is order isomorphic to τ, that is, there is a subsequence $\pi_{f(1)}, \pi_{f(2)}, \ldots, \pi_{f(m)}$, where $1 \leq f(1) < f(2) < \cdots < f(m) \leq n$, such that for all $1 \leq i, j \leq m$, we have $\pi_{f(i)} < \pi_{f(j)}$ if and only if $\tau_i < \tau_j$ and $\pi_{f(i)} > \pi_{f(j)}$ if and only if $\tau_i > \tau_j$. Otherwise, the ascent sequence π is said to *avoid* the pattern τ. For example, the ascent sequence 0120311252 has three occurrences of the pattern 100, namely, the subsequences 211, 311, and 322, but avoids the pattern 210. Note that within an occurrence of a pattern τ, letters corresponding to equal letters in τ must be equal within the occurrence.

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Following [7], we will write patterns for ascent sequences using non-negative integers, though patterns for other structures like permutations have traditionally been written with positive integers, to be consistent with the usual notation for ascent sequences which contains 0's. Thus, the traditional patterns will have different names here; for example 123 becomes 012 and 221 becomes 110.

If τ is a pattern, then let $\mathcal{S}_\tau(n)$ denote the set of ascent sequences of length n that avoid τ and $A_\tau(n)$ the number of such sequences. The set of *right-to-left maxima* in a sequence of numbers $a_1a_2\cdots a_n$ is the set of a_i such that $a_i > a_j$ for all $j > i$. Let RLmax(x) be the number of right-to-left maxima in a sequence x. Recall that the Catalan numbers are given by $C_n = \frac{1}{n+1}\binom{2n}{n}$ and that the Narayana numbers given by $N_{n,k} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$, $1 \le k \le n$, refine the Catalan numbers in that $C_n = \sum_{k=1}^{n} N_{n,k}$. It is well known that the number of 132-avoiding permutations of length n having exactly k ascents is given by $N_{n,k+1}$.

Let fwd(x) be the length of the maximal final weakly decreasing sequence in an ascent sequence x. For example, fwd$(010013014364332) = 5$ since 64332 has length 5. We also let zeros(x) denote the number of 0's in an ascent sequence x.

We now state the conjectures from [7] which we prove in the following sections.

Conjecture 3.2 (*[7]*). *We have $A_{0012}(n) = C_n$, the n-th Catalan number. Moreover, the bistatistic* (asc, fwd) *on $\mathcal{S}_{0012}(n)$ has the same distribution as* (asc, RLmax) *does on permutations avoiding the pattern* 132. *In particular, this implies that the number of ascents has the Narayana distribution on $\mathcal{S}_{0012}(n)$. Also, the bistatistics* (asc, fwd) *and* (asc, zeros) *have the same distribution on* 0012*-avoiding ascent sequences.*

Conjecture 3.4 (*[7]*). *The number $A_{0123}(n)$ equals the number of Dyck paths of semilength n and height at most* 5. *See sequence* A080937 *in* [13].

See Corollary 2.5 and Theorems 2.6 and 3.3. We also determine $A_{1012}(n)$ and thus show part of the following conjecture; see Theorem 3.2.

Conjecture 3.5 (*[7]*). *The patterns* 0021 *and* 1012 *are Wilf equivalent, and $A_{0021}(n) = A_{1012}(n)$ is given by the binomial transform of the Catalan numbers, which is sequence* A007317 *in* [13].

We remark that in our proof of Conjecture 3.2 in the next section, we first consider the joint distribution (asc, zeros, fwd) on the members of $\mathcal{S}_{0012}(n)$ not ending in 0 and determine a functional equation satisfied by its generating function which we denote by $g(x, y; u, v)$. Using the *kernel method* [1], we are then able to show $g(x, y; 1, u) = g(x, y; u, 1)$, which implies the equidistribution of (asc, fwd) and (asc, zeros) on $\mathcal{S}_{0012}(n)$. Comparison with the generating function for the distribution of (asc, RLmax) on 132-avoiding permutations then gives the remaining part of Conjecture 3.2. Furthermore, an expression for the generating function $g(x, y; u, v)$ may be recovered and the full distribution for (asc, zeros, fwd) can be obtained by extracting the coefficient of x^n from it.

In Section 3, we enumerate $\mathcal{S}_\tau(n)$ when $\tau = 1012$ and $\tau = 0123$, in the former case, making use of the kernel method. In this proof, we first describe refinements of the numbers $A_{1012}(n)$ by introducing appropriate auxiliary statistics on $\mathcal{S}_{1012}(n)$ and then write recurrences for these refined numbers. The recurrences may then be expressed as a functional equation which may be solved using the kernel method. See [16] for a further description and examples of the strategy of refinement in determining an explicit formula for a sequence. We conclude with a discussion of the case of avoiding 210 by ascent sequences and determine two related recurrences.

2. Distribution of some statistics on $\mathcal{S}_{0012}(n)$

In order to determine the distributions of some statistics on $\mathcal{S}_{0012}(n)$, we first refine the set as follows. Given $n \ge 1$, $0 \le m \le n-1$, and $1 \le r, \ell \le n-m$, let $A_{n,m,r,\ell}$ denote the subset of $\mathcal{S}_{0012}(n)$ whose members have m ascents, r zeros, and fwd value ℓ. For example, $\pi = 012334004332 \in A_{12,5,3,4}$. Let $a_{n,m,r,\ell} = |A_{n,m,r,\ell}|$; note that $\sum_{m,r,\ell} a_{n,m,r,\ell} = A_{0012}(n)$ for all n.

In what follows, it will be more convenient to deal with the members of $\mathcal{S}_{0012}(n)$ that do not end in a 0. Let $B_{n,m,r,\ell}$ denote the subset of $A_{n,m,r,\ell}$ whose members do not end in a 0 and let $b_{n,m,r,\ell} = |B_{n,m,r,\ell}|$. The array $b_{n,m,r,\ell}$ may be determined as described in the following lemma.

Lemma 2.1. *The array $b_{n,m,r,\ell}$ may assume non-zero values only when $n \ge 2$, $1 \le m \le n-1$, $1 \le r \le n-m$, and $1 \le \ell \le n-m$. It is determined for $n \ge 3$ by the recurrences*

$$b_{n,m,1,\ell} = \sum_{i=1}^{n-m}\sum_{j=0}^{t} b_{n-j-1,m-1,i-j,\ell-j}, \quad m \ge 2, \tag{1}$$

where $t = \min\{i-1, \ell-1\}$, and

$$b_{n,m,r,\ell} = b_{n-1,m,r-1,\ell} + \sum_{j=\ell+1}^{n-m} b_{n-1,m-1,r-1,j}, \quad m \ge 2 \text{ and } r \ge 2, \tag{2}$$