Discrete Mathematics 315–316 (2014) 53–64

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Extremal measures and clockwise overlays☆

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ARTICLE INFO

Article history:
Received 9 May 2012
Received in revised form 4 October 2013
Accepted 6 October 2013
Available online 29 October 2013

Keywords:
Horn inequality
Littlewood–Richardson rule
Clockwise overlay
Rigid tree measure

ABSTRACT

We establish in full generality the correspondence between saturated Horn inequalities and clockwise overlays. This was known in generic cases by work of Knutson, Tao and Woodward. We also prove that an extremal rigid measure forms a clockwise overlay with itself. Finally, we provide a simple test for the rigidity of a measure derived from the immersion of a tree.

1. Introduction

The honeycomb formulation of the Littlewood–Richardson rule [5] leads to the consideration of a special class of measures [1] defined in the plane. In this paper we consider three distinct but related questions arising in the study of these measures.

The first question concerns the concept of a *clockwise overlay*. We recall from [4] that a measure μ, all of whose densities are integers is naturally associated to a triple (A, B, C) of Hermitian matrices such that $A + B + C = 0$. The eigenvalues of these matrices are subject to a system of linear inequalities first proposed by Horn [3]. If this triple saturates one of the Horn inequalities, it was shown in [5] that μ can, at least generically, be written as a sum $\mu = \mu_1 + \mu_2$, where the pair (μ_1, μ_2) is a clockwise overlay. Roughly (and generically) speaking, this means that the supports of the measures μ_1 and μ_2 only intersect transversally and satisfy a certain orientation condition at the points of intersection. The case of general measures μ was not considered in [5] and in fact clockwise overlays were not defined in nongeneric situations. A more general notion of a clockwise overlay was introduced in [1]. The first result of this paper, Theorem 2.5, can be paraphrased as an extension of the cited result in [5], along with its converse.

Theorem 1.1. *A triple (A, B, C) associated to a measure μ saturates a Horn inequality if and only if $\mu = \mu_1 + \mu_2$ and the pair (μ_1, μ_2) is a clockwise overlay as defined in* [1].

A related result, Theorem 2.6, states that the reflection of a clockwise overlay in a horizontal line is a counterclockwise overlay. This is quite obvious in the generic case considered in [5]. Indeed, reflections change the orientation at transversal

☆ L. T. was supported in part by an NSF REU program at Indiana University. H. B. and W.S. L. were supported in part by grants from the National Science Foundation.

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0012-365X/$ – see front matter
http://dx.doi.org/10.1016/j.disc.2013.10.004

intersections of the supports. In the general case there may be no transversal intersections and new combinatorial arguments must be used.

The collection of all measures μ is a cone, and the extremal structure of this cone is essential in setting up explicit formulas for solving intersection problems in Grassmann manifolds, as seen in [1]. A special type of extremal measure is a *rigid tree measure*, obtained from an immersion of a binary tree into the plane. The second question we consider concerns the Horn inequalities for a triple (A, B, C) of Hermitian matrices associated with a rigid tree measure. Theorem 3.1 can be reformulated as follows.

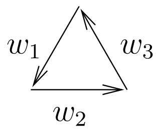
Theorem 1.2. *Assume that μ is a rigid tree measure, and (A, B, C) is the associated triple of Hermitian matrices. Then all the Horn inequalities for (A, B, C) are strict but $(A \oplus A, B \oplus B, C \oplus C)$ saturates some Horn inequality.*

The number of saturated Horn inequalities is at least three, and it is usually quite substantial. This result is also true for some nonrigid measures μ, but not for all of them.

The last question we consider is about the construction of rigid tree measures. It is very easy to construct measures which arise from immersions of trees, but it seems quite difficult to decide when a measure constructed this way is rigid. Theorem 4.1 is a test for identifying these rigid immersions. In order to obtain a rigid measure one must, roughly speaking, make sure that branch points are of only two types, and that a consistent orientation is preserved. The test is easily applied. It may not be effective if a complete enumeration of rigid tree measures (of fixed weight, for instance) is desired. Such enumerations are best done by considering the duals of rigid tree measures. We plan to return to this question in the future.

2. Measures, puzzles, clockwise overlays, and saturation

Our main object of study is a special class of measures in the plane, which we now define. Start with three vectors w_1, w_2, w_3 of unit length in $\mathbb{R}^2$ such that $w_1 + w_2 + w_3 = 0$.



The measures μ we are interested in are supported in a finite union of lines parallel to one of these three vectors, and satisfy the following two conditions.

(1) On each segment which does not intersect other segments in its support, μ is proportional to length; the constant of proportionality is the *density* of μ on that segment. The density of μ will be considered to be zero on segments outside its support.

(2) For any point $A \in \mathbb{R}^2$, we have

$$\delta_1^+(\mu, A) - \delta_1^-(\mu, A) = \delta_2^+(\mu, A) - \delta_2^-(\mu, A) = \delta_3^+(\mu, A) - \delta_3^-(\mu, A),$$

where $\delta_j^{\pm}(\mu, A)$ is the density of μ on the segment $\{A \pm t w_j : t \in (0, \varepsilon)\}$ for small ε.

Condition (2) implies that there are no points A for which exactly one of the densities $\delta_j^{\pm}$ is different from zero, and it is otherwise relevant only for the (finitely many) points for which at least three of these numbers are different from zero. These are called *branch points* of the measure μ. When μ has no branch points, its support consists of one or more parallel lines, on each of which the density is constant. We will generally assume the following condition.

(3) The measure μ has at least one branch point.

Under this additional assumption, the support of μ must be connected, and it will contain several half lines, each one pointing in the direction of one of the vectors $\pm w_j$. We denote by $\mathcal{M}$ the class of measures μ satisfying (1)–(3) and

(4) each half line in the support of μ points in the direction of w_1, w_2, or w_3.

Analogously $\mathcal{M}^*$ consists of those measures μ which satisfy (1)–(3) and

(4*) each half line in the support of μ points in the direction of $-w_1$, $-w_2$, or $-w_3$.

Clearly $\mathcal{M}$ and $\mathcal{M}^*$ are convex cones, and $\mathcal{M} \cap \mathcal{M}^* = \{0\}$.

Assume now that r is a positive number, and denote by $\triangle_r$ the closed triangle with vertices $0, r w_1$, and $r(w_1 + w_2)$. The collection of measures $\mu \in \mathcal{M}$, all of whose branch points belong to $\triangle_r$, is denoted $\mathcal{M}_r$. The class $\mathcal{M}_r^* \subset \mathcal{M}^*$ is defined analogously.

Consider a measure $\mu \in \mathcal{M}_r$, and a point $A \in \partial \triangle_r$. Assume that A belongs to the side of $\triangle_r$ parallel to w_{j+1}, where the addition is done modulo 3. We say that A is an *exit point* of μ if μ assigns positive density to the half line $\{A + t w_j : t \geq 0\}$. The corresponding density is called an *exit density* of μ. The measure μ is said to be *rigid* if there is no other measure $\nu \in \mathcal{M}_r$ which has the same exit points and exit densities as μ.

The preceding definitions are adapted by symmetry to measures in $\mathcal{M}_r^*$.