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Codes associated with the odd graphs

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ABSTRACT

Linear codes arising from the row span over any prime field $\mathbb{F}_p$ of the incidence matrices of the odd graphs $\mathcal{O}_k$ for $k \geq 2$ are examined and all the main parameters obtained. A study of the hulls of these codes for $p = 2$ yielded that for $\mathcal{O}_2$ (the Petersen graph), the dual of the binary hull from an incidence matrix is the binary code from points and lines of the projective geometry $PG_3(\mathbb{F}_2)$, which leads to a correspondence between the edges and vertices of $\mathcal{O}_2$ with the points and a collection of ten lines of $PG_3(\mathbb{F}_2)$, consistent with the codes.

The study also gives the dimension, the minimum weight, and the nature of the minimum words, of the binary codes from adjacency matrices of the line graphs $L(\mathcal{O}_k)$.

1. Introduction

Recent investigations of the codes from the $|V| \times |E|$ incidence matrices of k-regular connected graph $\Gamma = (V, E)$ in, for example [8,18,13,21,22], yielded observations that led to a more general approach for this study, using edge-connectivity of graphs, in [6,5]. This showed that, under certain very broad conditions on Γ, the codes over any field $\mathbb{F}_p$ from an incidence matrix G have the properties that: the dimension is $|V|$ or $|V| - 1$; the minimum weight is k and the words of weight k are the scalar multiples of the rows of G; there are no words of weight i such that $k < i < 2k-2$; the words of weight $2k-2$ are the scalar multiples of the differences of two rows of G corresponding to adjacent vertices. Thus the graph can be retrieved from the code. Such properties are reminiscent of the codes from finite projective planes: see [1, Chapter 6]. Codes from the adjacency matrices of graphs do not behave in such a uniform way. However, since for $p = 2$, $G^T G$ is an adjacency matrix for the line graph of Γ, $L(\Gamma)$, in such cases we can use the facts about the code from the incidence matrix for Γ for information about the binary code from the adjacency matrix of $L(\Gamma)$, including the dimension and minimum weight. In particular, the codes will not be trivial.

In this paper we examine these codes from incidence matrices of the odd graphs $\mathcal{O}_k$, and deduce properties of the binary codes from adjacency matrices of the line graphs $L(\mathcal{O}_k)$ and also the hulls of these codes, where the hull of a code C is $C \cap C^{\perp}$. The odd graphs $\mathcal{O}_k$ for $k \geq 2$ are the uniform subset graphs $\Gamma(2k+1, k, 0)$ whose vertices are the subsets of size k of a set of size $2k+1$, with two vertices being adjacent if the two k-subsets intersect in the empty set.[1] They are thus $(k+1)$-regular graphs. Binary codes from the adjacency matrices of these graphs were examined in [10, Chapter 6]. Here we consider p-ary codes from incidence matrices for these graphs, along with binary codes from the adjacency matrices of their line graphs, and the hulls of these.

Our main results are collected in the following theorem, where we use the notation that if A is a matrix then $C_p(A)$ denotes the row span of A over the prime field $\mathbb{F}_p$:

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[1] Frequently denoted by O_{k+1} in the literature.

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Theorem 1. *For $k \geq 2$, let $G_k = [g_{i,j}]$ be a $\binom{2k+1}{k} \times \frac{k+1}{2}\binom{2k+1}{k}$ incidence matrix for the odd graph $\mathcal{O}_k$, and let L_k be an adjacency matrix for the line graph $L(\mathcal{O}_k)$. For p any prime, let $\varepsilon_p = 0$ if p is odd, $\varepsilon_2 = 1$. Then:*

1. *For any prime p, $C_p(G_k)$ is a $\left[\frac{k+1}{2}\binom{2k+1}{k}, \binom{2k+1}{k} - \varepsilon_p, k+1\right]_p$ code.*
   If $k \geq 3$, the minimum words are the scalar multiples of the rows of G_k, there are no words of weight i where $k+1 < i < 2k$, and the words of weight $2k$ are the scalar multiples of the differences of two rows corresponding to two adjacent vertices.
   If $p = 2$, the same is true for $k = 2$. For p odd, $C_p(G_2)$ has more words of weight 3.
2. *If $E(G_k) = \langle g_{i,j} - g_{i,m} \mid 1 \leq i \leq 2k+1 \rangle$ over $\mathbb{F}_2$, then $E(G_k) = C_2(L_k)$. If $k = 2^l - 1$ for some $l \geq 2$, then $C_2(L_k) = C_2(G_k)$; otherwise $C_2(L_k)$ has codimension 1 in $C_2(G_k)$ and is a $\left[\frac{k+1}{2}\binom{2k+1}{k}, \binom{2k+1}{k} - 2, 2k\right]_2$ code, with the words of weight $2k$ the rows of L_k.*
3. *For all $k \geq 2$, $\mathrm{Hull}(C_2(G_k))$ and $\mathrm{Hull}(C_2(L_k))$ have minimum weight at least $2k+2$, and either they are equal or one has codimension 1 in the other. For k even, $\dim(\mathrm{Hull}(C_2(G_k))) = \binom{2k-1}{k} + 2^{k-1} - 1$; for k odd, $\dim(\mathrm{Hull}(C_2(G_k))) = \binom{2k}{k-1} - 1$.*
   Further, for the strongly regular $(10, 3, 0, 1)$ Petersen graph $\mathcal{O}_2$,
   $$(\mathrm{Hull}(C_2(G_2)))^{\perp} = C_2(G_2) + C_2(G_2)^{\perp} = C_2(PG_{3,1}(\mathbb{F}_2)) = \mathcal{H}_4,$$
   where $\mathcal{H}_r$ denotes the Hamming code of length $2^r - 1$, $\langle \mathrm{Hull}(C_2(G_2)), \boldsymbol{\jmath}_{15} \rangle = C_2(PG_{3,2}(\mathbb{F}_2))$. There is a correspondence between the edges and vertices of $\mathcal{O}_2$ and the 15 points and a set of ten lines of $PG_3(\mathbb{F}_2)$, consistent with the codewords. The edges of the 15 8-cycles of $\mathcal{O}_2$ are the supports of the non-zero words of $\mathcal{H}_4^{\perp}$, with complements the 15 Fano planes $PG_2(\mathbb{F}_2)$ in $PG_3(\mathbb{F}_2)$.

General terminology is given in Section 2. The results collected in the theorem appear as propositions in Sections 3–6. Some further general results about binary codes of adjacency matrices of line graphs and their hulls are shown in Sections 4 and 5. The results for the Petersen graph are in Section 6. This is followed by step-by-step procedures to correspond the points of $PG_3(\mathbb{F}_2)$ with the edges of $\mathcal{O}_2$, and the converse operation of obtaining the graph from the points and a set of ten lines of $PG_3(\mathbb{F}_2)$. Section 7 concerns the use of these codes for permutation decoding.

2. Background, terminology, and previous results

2.1. Designs and codes

The notation for designs and codes is as in [1]. An incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{J})$, with point set $\mathcal{P}$, block set $\mathcal{B}$ and incidence $\mathcal{J}$ is a t-(v, k, λ) design if $|\mathcal{P}| = v$, every block $B \in \mathcal{B}$ is incident with precisely k points, and every t distinct points are together incident with precisely λ blocks. A design is *symmetric* if it has the same number of points as blocks. The *code $C_F(\mathcal{D})$ of the design $\mathcal{D}$* over the finite field F is the space spanned by the incidence vectors of the blocks over F. If $\mathcal{Q} \subseteq \mathcal{P}$, then we will denote the *incidence vector* of $\mathcal{Q}$ by $v^{\mathcal{Q}}$. Thus $C_F(\mathcal{D}) = \left\langle v^B \mid B \in \mathcal{B} \right\rangle$, and is a subspace of $F^{\mathcal{P}}$. For any $w \in F^{\mathcal{P}}$ and $P \in \mathcal{P}$, $w(P)$ denotes the value of w at P. If $F = \mathbb{F}_p$ then the *p-rank* of $\mathcal{D}$, written $\mathrm{rank}_p(\mathcal{D})$, is the dimension of $C_p(\mathcal{D})$, writing $C_p(\mathcal{D})$ for $C_F(\mathcal{D})$.

All the codes here are *linear codes*, and the notation $[n, k, d]_q$ is used for a q-ary code C of length n, dimension k, and minimum weight d, where the *weight wt* (v) of a vector v is the number of non-zero coordinate entries. The *support*, $\mathrm{Supp}(v)$, of a vector v is the set of coordinate positions where the entry in v is non-zero. A *generator matrix* for C is a $k \times n$ matrix with rows a basis for C, and the *dual* code $C^{\perp}$ is the orthogonal under the standard inner product $(,)$, i.e. $C^{\perp} = \{v \in F^n \mid (v, c) = 0$ for all $c \in C\}$. If $C = C_p(\mathcal{D})$, where $\mathcal{D}$ is a design, then $C \cap C^{\perp}$ is the *hull* of $\mathcal{D}$ or C. A *check matrix* for C is a generator matrix for $C^{\perp}$. The *all-one vector* will be denoted by $\boldsymbol{\jmath}$, and is the vector with all entries equal to 1. The all-one vector of length m is written $\boldsymbol{\jmath_m}$. We call two linear codes *isomorphic* if they can be obtained from one another by permuting the coordinate positions. An *automorphism* of a code C is an isomorphism from C to C. The automorphism group will be denoted by $\mathrm{Aut}(C)$. An *information set* for C is the set of k coordinate positions of a set of k linearly independent columns of a generator matrix for C. The remaining coordinates are called a *check set*.

For any finite field $\mathbb{F}_q$ of order q, the set of points and r-dimensional subspaces of an m-dimensional projective geometry forms a 2-design which we will denote by $PG_{m,r}(\mathbb{F}_q)$. The *automorphism group* of each of these designs is the full projective semi-linear group, $P\Gamma L_{m+1}(\mathbb{F}_q)$ and is 2-transitive on points. The codes of these designs are subfield subcodes of the generalised Reed–Muller codes: see [1, Chapter 5] for a full treatment.

2.2. Graphs and codes

The *graphs*, $\Gamma = (V, E)$ with vertex set V and edge set E, are simple. If $X, Y \in V$ and X and Y are adjacent, we write $X \sim Y$, and XY or $[X, Y]$ for the *edge* in E that they define. The *set of neighbours* of $X \in V$ is denoted by $\mathcal{N}(X)$, and the *valency of* X is $|\mathcal{N}(X)|$. Γ is *regular* if all the vertices have the same valency. A *path* of length r from vertex X to vertex Y is a sequence X_i, for $0 \leq i \leq r-1$, of distinct vertices with $X = X_0$, $Y = X_{r-1}$, and $X_{i-1} \sim X_i$ for $1 \leq i \leq r-1$. It is *closed* of length r if $X \sim Y$, in which case we write it $(X_0, \ldots, X_{r-1})$. The graph is *connected* if there is a path between any two vertices. A *perfect matching* is a set S of disjoint edges such that every vertex is on exactly one member of S. An *adjacency matrix A* is a $|V| \times |V|$ matrix with entries a_{ij} such that $a_{ij} = 1$ if vertices X_i and X_j are adjacent, and $a_{ij} = 0$ otherwise. An *incidence matrix*