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Critical sets for Sudoku and general graph colorings

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ABSTRACT

We discuss the problem of finding *critical sets* in graphs, a concept which has appeared in a number of guises in the combinatorics and graph theory literature. The case of the Sudoku graph receives particular attention, because critical sets correspond to minimal fair puzzles. We define four parameters associated with the sizes of extremal critical sets and (a) prove several general results concerning the properties of these parameters, including their computational intractability, (b) compute their values exactly for some classes of graphs, (c) obtain bounds for generalized Sudoku graphs, and (d) offer a number of open questions regarding critical sets and the aforementioned parameters.

A recent announcement due to McGuire et al. [9] surprised many in the community of Sudoku researchers, amateur and professional alike, with its complete resolution of the "minimum number of clues" (MNC) problem. *Sudoku* is a single-player game in which one completes a partial 9×9 matrix M all of whose entries are drawn from $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ by appealing to the *rules*: no number may appear twice in any row, any column, or any of the nine "blocks", each a 3×3 submatrix with indices $\{3a+1, 3a+2, 3a+3\} \times \{3b+1, 3b+2, 3b+3\}$ for $a, b \in \{0, 1, 2\}$. A "board" is a matrix adhering to these rules; a "puzzle" is a partially filled-in board; the nonempty entries of a puzzle are called "clues" or "givens". A puzzle is said to be "fair" if it can be completed to a valid board in precisely one way. The MNC problem asks: what is the lowest number of clues in a fair Sudoku puzzle? While it was long suspected that the answer is 17, a proof seemed out of reach until [9].

However, the sense in which the authors of [9] "proved" that the solution is indeed 17 arguably does not meet modern standards of mathematical rigor. (The paper briefly acknowledges this deficiency, although the wide popular press reporting of the result typically did not address this important, if subtle, issue. See [7] for more discussion of philosophical aspects of computer-assisted proof.) There are several interesting ideas presented in the aforementioned manuscript — mostly careful case reductions and very clever search strategies — but, in the end, the result relied on a *year-long* computation, amounting to 7.1 million core hours on an SGI Altic ICE 8200EX cluster with 320 nodes, each of which consisted of two Intel Xeon E5650 hex-core processors with 24 GB of RAM. Even if one sets aside well-worn (and important) philosophical critiques of computer-assisted proofs that appeal to uncheckability by humans and the social nature of proof, it is almost inconceivable that this enormous computation on an extremely complicated configuration of networked and nested devices did not experience hardware errors (due to manufacturing defects, cosmic rays, background radiation, the inherent stochasticity of quantum mechanics, etc.) *and* software errors (bugs in the various operating systems, firmware, algorithmic code, GUIs, etc.). While such errors may not have produced an incorrect answer, they certainly undermine the definitiveness of the result. Therefore, we wish to draw attention to the subject matter of "critical sets" for graph colorings, a concept that neatly generalizes the MNC problem as well as several other questions scattered throughout the discrete mathematics literature, in the hope that greater visibility might eventually lead to human-readable solutions to questions like that of the MNC.

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We begin by defining "determining sets" for graph colorings: a set S of vertices with the property that the coloring, restricted to S, can be completed in precisely one way (i.e., back to the original coloring).

Definition 1. A "determining set" of vertices in a graph $G = (V, E)$ with respect to a proper vertex coloring $c : V \to [\chi(G)]$ is a set $S \subseteq V$ with the property that, for any proper vertex coloring c' of G, if $c'|_S \equiv c|_S$, then $c' \equiv c$ on all of V.

If a determining set is minimal with respect to this property, we call it "critical".

Definition 2. A "critical set" of vertices in a graph $G = (V, E)$ with respect to a proper vertex coloring $c : V \to [\chi(G)]$ is a minimal determining set for the pair (G, c).

The cardinalities of the largest and smallest critical sets in various graphs have appeared in a number of guises. Latin squares (and, of course, the special subclass of them that comprises Sudoku boards), matching theory, design theory, and the study of dominating vertex sets all feature variants of this idea. (See [5] for a more comprehensive list of related topics and references.) Therefore, we define the following parameters.

Definition 3. For a graph $G = (V, E)$ and a vertex coloring $c : V \to [\chi(G)]$, define

$$\mathrm{scs}(G, c) = \min\{|X| : X \text{ is a critical set for } (G, c)\}$$

and

$$\mathrm{LCS}(G, c) = \max\{|X| : X \text{ is a critical set for } (G, c)\}.$$

Definition 4. For a graph $G = (V, E)$, define

$$\underline{\mathrm{scs}}(G) = \min_{\substack{c:V(G)\to[\chi(G)]\\ c \text{ proper}}} \mathrm{scs}(G, c)$$

$$\underline{\mathrm{LCS}}(G) = \min_{\substack{c:V(G)\to[\chi(G)]\\ c \text{ proper}}} \mathrm{LCS}(G, c).$$

Similarly, define

$$\overline{\mathrm{scs}}(G) = \max_{\substack{c:V(G)\to[\chi(G)]\\ c \text{ proper}}} \mathrm{scs}(G, c)$$

$$\overline{\mathrm{LCS}}(G) = \max_{\substack{c:V(G)\to[\chi(G)]\\ c \text{ proper}}} \mathrm{LCS}(G, c).$$

A few words on notation. For graph-theoretic concepts, unless stated explicitly, we generally rely on the conventions of [3]; in particular, we sometimes write the edge $\{x, y\}$ simply as xy. For a vertex $v \in V(G)$, we define the "neighborhood" of v to be $N_G(v) = \{w \in V(G) : vw \in E(G)\}$, where the subscript may be omitted if it is obvious. The symbol "$\square$" denotes the Cartesian product, i.e., given two graphs $G = (U, E)$ and $H = (V, F)$, the vertex set of $G\square H$ is $U \times V$, and its edge set is given by all pairs of the form $\{(u, v), (u', v')\}$ with $[(u = u') \wedge (vv' \in E(H))] \vee [(v = v') \wedge (uu' \in E(G))]$. We denote the set $\{1, \dots, n\}$ by $[n]$. Finally, let Sud_n denote the Sudoku graph, i.e., $V(\mathrm{Sud}_n) = [n^2] \times [n^2]$ and $\{(a, b), (c, d)\} \in E(\mathrm{Sud}_n)$ for $(a, b) \neq (c, d)$ iff $a = c$, $b = d$, or $(\lceil a/n\rceil = \lceil c/n\rceil) \wedge (\lceil b/n\rceil = \lceil d/n\rceil)$.

1. General bounds and observations

In this section, we introduce a few important definitions and prove some general facts concerning critical sets.

Definition 5. A coloring $c : V(G) \to [k]$ of a graph G is "optimal" if $k = \chi(G)$.

Definition 6. A graph G is "uniquely colorable" if it has exactly one optimal coloring up to permutation of the colors.

Definition 7. A graph G is "critically k-uniform" if every critical set $S \subset V(G)$ satisfies $|S| = k$; it is "critically uniform" if it is critically k-uniform for some k.

Note that the condition of G being critically k-uniform is equivalent to the statement that

$$k = \overline{\mathrm{LCS}}(G) = \underline{\mathrm{LCS}}(G) = \overline{\mathrm{scs}}(G) = \underline{\mathrm{scs}}(G).$$

Proposition 1. *If a graph G is uniquely colorable, then G is critically $(\chi(G) - 1)$-uniform.*

Proof. First, every critical set S must have cardinality at least $\chi(G) - 1$; if $|S| < \chi(G) - 1$, then, given any coloring $c : V(G) \to [\chi(G)]$, there are at least two extensions of $c|_S$ to a proper coloring: c itself and c', where

$$c'(v) = \begin{cases} c(v) & \text{if } c(v) \in c(S) \\ \pi(c(v)) & \text{if } c(v) \notin c(S), \end{cases}$$

where π is any permutation of $[\chi(G)]$ such that $\pi|_{[\chi(G)]\setminus c(S)}$ is not the identity function.