Discrete Mathematics 315–316 (2014) 128–134

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Every 3-polytope with minimum degree 5 has a 6-cycle with maximum degree at most 11

O.V. Borodin [a,b], A.O. Ivanova [c,*], A.V. Kostochka [d,a]

[a] Sobolev Institute of Mathematics, Novosibirsk 630090, Russia
[b] Novosibirsk State University, Novosibirsk 630090, Russia
[c] Ammosov North-Eastern Federal University, Yakutsk, 677891, Russia
[d] University of Illinois at Urbana–Champaign, Urbana, IL 61801, USA

ARTICLE INFO

Article history:
Received 2 July 2013
Received in revised form 15 October 2013
Accepted 21 October 2013
Available online 7 November 2013

Keywords:
Planar graph
Plane map
Structure properties
3-polytope
Weight

ABSTRACT

Let $\varphi_P(C_6)$ (respectively, $\varphi_T(C_6)$) be the minimum integer k with the property that every 3-polytope (respectively, every plane triangulation) with minimum degree 5 has a 6-cycle with all vertices of degree at most k. In 1999, S. Jendrol' and T. Madaras proved that $10 \le \varphi_T(C_6) \le 11$. It is also known, due to B. Mohar, R. Škrekovski and H.-J. Voss (2003), that $\varphi_P(C_6) \le 107$.

We prove that $\varphi_P(C_6) = \varphi_T(C_6) = 11$.

1. Introduction

The *degree* $d(x)$ of a vertex or face x in a plane graph G is the number of incident edges. A *k-vertex* (*k-face*) is a vertex (face) with degree k, a k^+*-vertex* has degree at least k, etc. The minimum vertex degree of G is $\delta(G)$. We will drop the arguments whenever this does not lead to confusion.

A normal plane map (NPM) is a plane pseudograph in which loops and multiple edges are allowed, but $d(x) \ge 3$ for every vertex or face x. As proved by Steinitz [20], the 3-connected plane graphs are planar representations of the convex three-dimensional polytopes, called hereafter 3-polytopes.

In this note, we consider the class $\mathbf{M_5}$ of NPMs with $\delta = 5$ and its subclasses $\mathbf{P_5}$ of 3-polytopes and $\mathbf{T_5}$ of plane triangulations. A cycle on k vertices is denoted by C_k, and S_k stands for a k-star centered at a 5-vertex. (So, S_k is a subgraph of M_5 on a 5-vertex and k vertices adjacent to it, where $0 \le k \le 5$.)

In 1904, Wernicke [21] proved that $M_5 \in \mathbf{M_5}$ implies, in M_5, the presence of a vertex of degree 5 adjacent to a vertex of degree at most 6. This result was strengthened by Franklin [8] in 1922 to the existence of a vertex of degree 5 with two neighbors of degree at most 6. In 1940, Lebesgue [15, p. 36] gave an approximate description of the neighborhoods of vertices of degree 5 in $\mathbf{T_5}$.

The *weight* $w_M(H)$ is the maximum over $M_5 \in \mathbf{M_5}$ of the minimum degree-sum of the vertices of H over subgraphs H of M_5. The *weights* $w_P(H)$ and $w_T(H)$ are defined similarly for $\mathbf{P_5}$ and $\mathbf{T_5}$, respectively.

The bounds $w_M(S_1) \le 11$ (Wernicke [21]) and $w_M(S_2) \le 17$ (Franklin [8]) are tight. It was proved by Lebesgue [15] that $w_M(S_3) \le 24$ and $w_M(S_4) \le 31$, which was improved much later to the following tight bounds: $w_M(S_3) \le 23$ (Jendrol'

* Corresponding author.
E-mail addresses: brdnoleg@math.nsc.ru (O.V. Borodin), shmgnanna@mail.ru (A.O. Ivanova), kostochk@math.uiuc.edu (A.V. Kostochka).

0012-365X/$ – see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.disc.2013.10.021

and Madaras [10]) and $w_M(S_4) \leq 30$ (Borodin and Woodall [6]). Note that $w_M(S_3) \leq 23$ readily implies $w_M(S_2) \leq 17$ and immediately follows from $w_M(S_4) \leq 30$ (it suffices to delete a vertex of maximum degree from a star of the minimum weight).

It follows from Lebesgue [15, p. 36] that $w_T(C_3) \leq 18$. In 1963, Kotzig [14] gave another proof of this fact and conjectured that $w_T(C_3) \leq 17$. (The bound 17 is easily shown to be tight.)

In 1989, Kotzig's conjecture was confirmed by Borodin [2] in a more general form, by proving $w_M(C_3) = 17$. Another consequence of this result is the confirming of a conjecture of Grünbaum [9] from 1975 that the cyclic connectivity (defined as the minimum number of edges to be deleted from a graph to obtain two components each containing a cycle) of every 5-connected planar graph is at most 11, which is tight (a bound of 13 was earlier obtained by Plummer [19]).

It also follows from Lebesgue [15, p. 36] that $w_T(C_4) \leq 26$ and $w_T(C_5) \leq 31$. In 1998, Borodin and Woodall [6] proved $w_T(C_4) = 25$ and $w_T(C_5) = 30$.

Now let $\varphi_M(H)$ ($\varphi_P(H)$, $\varphi_T(H)$) be the minimum integer k with the property that every normal plane map (3-polytope, plane triangulation) with minimum degree 5 has a copy of H with all vertices of degree at most k.

It follows from Franklin [8] that $\varphi_M(S_2) = 6$. From $w_M(C_3) = 17$ (Borodin [2]), we have $\varphi_M(C_3) = 7$. In 1996, Jendrol' and Madaras [10] proved $\varphi_M(S_4) = 10$ and $\varphi_T(C_4) = \varphi(C_5) = 10$. R. Soták (personal communication; see surveys of Jendrol' and Voss [12,13]) proved $\varphi_P(C_4) = 11$ and $\varphi_P(C_5) = 10$.

In 1999, Jendrol' et al. [11] obtained the following bounds: $10 \leq \varphi_T(C_6) \leq 11$, $15 \leq \varphi_T(C_7) \leq 17$, $15 \leq \varphi_T(C_8) \leq 29$, $19 \leq \varphi_T(C_9) \leq 41$, and $\varphi_T(C_p) = \infty$ whenever $p \geq 11$. Madaras and Soták [17] proved $20 \leq \varphi_T(C_{10}) \leq 415$.

For the broader class $\mathbf{P_5}$ (an easy induction proof shows that every planar triangulation on at least four vertices is 3-connected), it is known that $10 \leq \varphi_P(C_6) \leq 107$ due to Mohar et al. [18] (in fact, this bound is proved in [18] for all 3-polytopes with $\delta \geq 4$ in which no 4-vertex is adjacent to a 4-vertex), and $\varphi_P(C_7) \leq 359$ is due to Madaras et al. [16].

The purpose of our note is to prove that $\varphi_P(C_6) = \varphi_T(C_6) = 11$. This answers a question raised by Jendrol' et al. [11].

Theorem 1. *Every 3-polytope with minimum degree 5 has a 6-cycle such that each of its vertices has degree at most 11, and this bound is tight.*

Other structural results on $\mathbf{M_5}$, some of which have application to coloring, can be found in the papers already mentioned and in [3,4,7,16–18].

One of the ideas used in our proof is to look for a suitable 6-cycle not in the whole graph but in a carefully chosen portion of it. A similar approach to coloring problems on plane graphs is described in a survey [5, pp. 520–521], and it has been used by us several times, beginning with [1].

2. Proving the tightness of Theorem 1

We transform the octahedron (the 4-regular plane triangulation on six vertices) to a plane triangulation in which every 6-cycle goes through a vertex of degree at least 11, replacing each of the eight 3-faces of the octahedron by the configuration shown in Fig. 1.

More specifically, half of the image of every edge (partly invisible) of the octahedron starts at an "angular" 12-vertex, goes through an 11-vertex, cuts an edge between two 5-vertices, then goes through two 5-vertices, cuts another edge between two 5-vertices, and ends in a 12-vertex, the mid-point of the image of the edge. The graph obtained has only 5-, 11-, and 12-vertices. Furthermore, every 5-vertex belongs to a blue (shadowed) triangle. It is easily seen that the subgraph on 5-vertices does not contain 6-cycles.

Note that we could use instead of the octahedron any plane triangulation with $\delta \geq 4$ to obtain a plane triangulation with the desired property.

3. Proving the upper bound in Theorem 1

Suppose G' is a counterexample to the main statement of Theorem 1. Thus G' is a 3-polytope with $\delta = 5$ in which no 6-cycle avoids a 12^+-vertex.

By Euler's formula $|V'| - |E'| + |F'| = 2$ for G', we have

$$\sum_{v \in V'} (d(v) - 4) + \sum_{f \in F'} (d(f) - 4) = -8. \tag{1}$$

This implies that G' has a 3-face. So we may assume that the external face of G' is bounded by a 3-cycle with the vertex set T'.

A *special triangle* $T^* = t_1t_2t_3$ of G' is a 3-cycle of G' with the fewest vertices inside. We define G to be the subgraph of G' induced by the vertices inside T^*. The vertices of G are *internal*, and the vertices t_1, t_2, and t_3 are *special*.

By G^{**} we denote the subgraph of G' induced by the vertices of $G \cup T^*$. In particular, $T^* = T'$ when $G^{**} = G'$. In both cases, T^* is the boundary $\partial(f_\infty)$ of the external face f_∞ of G^{**}.