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Induced matchings in subcubic graphs without short cycles

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ABSTRACT

An induced matching of a graph G is a set M of edges of G such that every two vertices of G that are incident with distinct edges in M have distance at least 2 in G. The maximum number of edges in an induced matching of G is the induced matching number $\nu_s(G)$ of G. In contrast to ordinary matchings, induced matchings in graphs are algorithmically difficult. Next to hardness results and efficient algorithms for restricted graph classes, lower bounds are therefore of interest.

We show that if G is a connected graph of order $n(G)$, maximum degree at most 3, girth at least 6, and without a cycle of length 7, then $\nu_s(G) \geq \frac{n(G)-1}{5}$, and we characterize the graphs achieving equality in this lower bound.

1. Introduction

Some of the most important and beautiful results in graph theory concern matchings [15]. Tutte's characterization of graphs having a perfect matching [20] and Edmonds' maximum matching algorithm [8] are clearly examples for this claim. Vizing's theorem stating that the chromatic index of a graph, that is, the minimum number of matchings into which its edge set can be partitioned, is at most one more than its maximum degree [21] is another example. These seminal contributions and especially the very nice structural and algorithmic results related to matchings motivated the investigation of related concepts. In the present paper we consider induced matchings in graphs, which are one such concept. A set M of edges of a graph G is an *induced matching of* G if every two vertices of G that are incident with distinct edges in M have distance at least 2 in G.

The first to consider induced matchings were Stockmeyer and Vazirani [19] who proved computational hardness of the maximum induced matching problem. Their hardness result was strengthened in many ways and numerous graph classes where a maximum induced matching can be found efficiently were discovered [2,4,5,12,16]. The maximum numbers of edges in graphs of bounded maximum degree without a non-trivial induced matching [6] as well as in bipartite graphs without a large induced matching [10] have been determined. Heuristics that find large induced matchings in (random) cubic graphs [7], planar graphs without vertices having the same neighborhood [9], and subcubic planar graphs [14] have been investigated.

Further motivation to study induced matchings comes from a problem posed by Erdős and Nešetřil concerning the *strong chromatic index* of a graph defined as the minimum number of induced matchings into which its edge set can be partitioned. As a variant of Vizing's result, Erdős and Nešetřil conjectured [11] that the strong chromatic index of a graph G of maximum degree $\Delta(G)$ is at most $\frac{5}{4}\Delta(G)^2$. It was shown that the strong chromatic index of graphs of maximum degree at most 3 [1,13] and of bipartite graphs of maximum degree at most 3 [18] is bounded by 10 and 9, respectively. A simple greedy argument implies that the strong chromatic index of a graph G of maximum degree $\Delta(G)$ is at most $2\Delta(G)(\Delta(G)-1)+1$.

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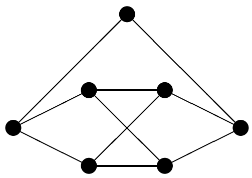

Fig. 1. A subcubic graph G_1 of girth 4 with $n(G_1) = 7$ and $\nu_s(G_1) = 1$.

Under stronger assumptions better bounds were obtained [3] and also for large maximum degrees [17] improvements are known. Numerous related results and problems concerning graphs of maximum degree 3 that are bipartite and/or planar and/or of large girth are discussed in [11].

Our contribution is a sharp lower bound on the largest size, denoted by $\nu_s(G)$, of an induced matching in a subcubic graph G without short cycles. The motivation for this result is our conjecture that

$$\nu_s(G) \geq \frac{n(G)}{6}$$

holds for every cubic graph G of order $n(G)$. This conjecture would be best possible; if B_0 denotes the unique tree of order 6 with two vertices of degree 3 and the cubic graph G arises from B_0 and four copies of the graph G_1 in Fig. 1 by identifying each of the four vertices of degree 1 with one of the four vertices of degree 2, then $\nu_s(G) = \frac{n(G)}{6}$. For cubic planar graphs, our conjecture follows from a result of Kang et al. [14].

In order to make the conjecture accessible to an inductive proof, we consider subcubic graphs instead of cubic graphs. Unfortunately, as illustrated in Fig. 1, there is a subcubic graph G_1 with $\nu_s(G_1) < \frac{n(G_1)}{6}$. Even when expressing the lower bound on $\nu_s(G)$ in terms of the number $m(G)$ of edges of G, this graph remains problematic; while we conjecture $\nu_s(G) \geq \frac{m(G)}{9}$ for a cubic graph G it only satisfies $\nu_s(G_1) = \frac{m(G_1)}{10}$. We exclude this problematic graph by a girth condition. Since G_1 is triangle-free, the girth has to be at least 5.

Our main result implies

$$\nu_s(G) \geq \frac{n(G) - 1}{5}$$

for a connected subcubic graph G of order $n(G)$ and girth at least 8. We believe that this bound already holds for girth at least 5.

2. Results

We consider only simple, finite, and undirected graphs and use standard terminology. Let G be a graph. Let $i(G)$ denote the number of isolated vertices of G. For a set M of edges of G, let $V(M)$ denote the set of vertices of G incident with an edge in M. A vertex u of G is *M-free* if $u \notin V(M)$ and it is *M-far* if $N_G[u] \cap V(M) = \emptyset$. Note that a matching M of G is *induced* if $V(M)$ induces a 1-regular subgraph of G. A vertex of degree 1 is an *end vertex*.

Recall that B_0 is the tree of order 6 with two vertices of degree 3 and four end vertices.

We now give a recursive definition of the family $\mathcal{B}$ of *bad graphs*. Furthermore, to each bad graph B we associate a set of its so-called *units*, which are edge-disjoint subgraphs of B and whose union is B:

- If the graph B is isomorphic to B_0, then B belongs to $\mathcal{B}$ and $\{B\}$ is the set of units of B.
- If B is a subcubic graph and B' as well as B'' are graphs in $\mathcal{B}$ such that B' is isomorphic to B_0 and B arises from the disjoint union of B' and B'' by identifying a vertex in B' and a vertex in B'', then B belongs to $\mathcal{B}$. The set of units of B is the union of the set $\{B'\}$ of units of B' and the set of units of B''.

Note that a graph B in $\mathcal{B}$ arises from the union of disjoint copies of B_0, which are the units of B, by identifying vertices in a tree-like way. It follows easily that a subcubic graph that arises from the disjoint union of two graphs B' and B'' in $\mathcal{B}$ by identifying a vertex in B' and a vertex in B'' belongs to $\mathcal{B}$. A unit U of a graph B in $\mathcal{B}$ is an *end unit* of B if at most one vertex of U has a larger degree in B than in U.

We collect some properties of bad graphs.

Lemma 1. *Let $B \in \mathcal{B}$. Let x and y be two not necessarily distinct vertices of B of degree at most 2.*

(i) *If B has p units, then $n(B) = 5p + 1$, B has at least $2p + 2$ end vertices, and $5\nu_s(B) = n(B) - 1$.*
(ii) *There is a maximum induced matching M of B such that one of x and y is M-free and the other one is M-far.*
(iii) *If x and y do not belong to the vertex set of some unit of B, then there is a maximum induced matching M of B such that x and y are both M-far.*

Proof. We prove all statements simultaneously by induction on the number p of units of B. If $p = 1$, then B is B_0 and (i) and (ii) are easily verified while (iii) is void and hence trivially true. Now let $p \geq 2$.