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Note

Clique partitions of complements of forests and bounded degree graphs

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Abstract

In this paper, we prove that for any forest $F \subset K_n$, the edges of $E(K_n)\backslash E(F)$ can be partitioned into $O(n \log n)$ cliques. This extends earlier results on clique partitions of the complement of a perfect matching and of a hamiltonian path in K_n.

In the second part of the paper, we show that for n sufficiently large and any $\varepsilon \in (0, 1]$, if a graph G has maximum degree $O(n^{1-\varepsilon})$, then the edges of $E(K_n)\backslash E(G)$ can be partitioned into $O(n^{2-(1/2)\varepsilon} \log^2 n)$ cliques provided there exist certain Steiner systems. Furthermore, we show that there are such graphs G for which $\Omega(\varepsilon^2 n^{2-2\varepsilon})$ cliques are required in every clique partition of $E(K_n)\backslash E(G)$.

Keywords: Clique partitions; Steiner systems; Probabilistic method

1. Introduction

A *clique partition* of a graph G is a collection of nontrivial complete subgraphs of G (called cliques) that partition the edge set of G. In this paper, we study the problem of finding clique partitions of $K_n\backslash F$, where $F \subset K_n$ is a forest or a graph of maximum degree Δ. Here $K_n\backslash F$ refers to the graph on $V(K_n)$ consisting of all edges of K_n which are not in F, and is called the *complement of* F. We denote by $\mathrm{cp}(K_n\backslash F)$ the *clique partition number* of $K_n\backslash F$, which is the smallest number of cliques partitioning $E(K_n)\backslash E(F)$. Any further notation not defined here is found in Bondy and Murty [4].

It was Orlin [8] who first asked about the asymptotics of the clique partition number of the complement of a perfect matching. Wallis [9] showed that for any fixed $\varepsilon > 0$, the clique partition number of the complement of a perfect matching is $o(n^{1+\varepsilon})$. Later, Gregory et al. [7] improved this result to $O(n \log \log n)$, and conjectured that it is asymptotic to n. Using the same argument as for the complement of a perfect matching, Wallis [10] proves that the same bound holds for the complement of a Hamiltonian path in K_n. Wallis [10] also notes that this technique can be extended to the complement of H, where every vertex of H is of degree one or two. In the case that F is a forest, we prove the following theorem:

Theorem 1. *Let $F \subset K_n$ be a forest. Then* $\mathrm{cp}(K_n\backslash F) = O(n \log n)$.

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Note that the forest F may not necessarily be a spanning forest of K_n. The proof of Theorem 1 will be given in Section 2. We are not aware of any lower bounds for $\mathrm{cp}(K_n \backslash F)$ which are not of order of magnitude $\mathrm{O}(n)$; we conjecture that there exist forests for which

$$\frac{\mathrm{cp}(K_n \backslash F)}{n} \to \infty.$$

In the second part of the paper, we are interested in estimating clique partition numbers of dense graphs—we wish to find bounds on $\mathrm{cp}(K_n \backslash G)$ when G has maximum degree Δ. An early result of Erdős and de Bruijn [5] shows that $\mathrm{cp}(K_n \backslash K_2) = n-1$ for $n \geqslant 3$, with equality only for clique partitions consisting of $n-2$ complete graphs of order two and one complete graph of order $n-2$. Using projective planes, Wallis [10] showed if H is a graph with at most $\sqrt{n}$ vertices, then $K_n \backslash H$ can be partitioned into $\mathrm{O}(n)$ cliques. A projective plane is a particular example of a family of sets called a Steiner system. We recall that a *Steiner* (n, k)*-system* is a family of k-element subsets of an n-element set such that each pair of elements appears in exactly one of the k-element sets. In other words, a Steiner (n, k)-system provides a clique partition of K_n into cliques of size k; in particular $\binom{n}{2} / \binom{k}{2}$ cliques are present in this partition. Conditional on the existence of Steiner systems with certain parameters, we give bounds on $\mathrm{cp}(K_n \backslash G)$ for graphs G with asymptotically prescribed maximum degree $\Delta(G)$:

Theorem 2. *Let G be a graph on n vertices, let $k = \lfloor (\frac{n}{2\Delta})^{1/2} \rfloor$, and suppose there exists a Steiner (n, k)-system. Then, provided that $\Delta(G) = \mathrm{o}(\frac{n}{\log^4 n})$ as n tends to infinity,*

$$\mathrm{cp}(K_n \backslash G) = \mathrm{O}(n^{3/2} \Delta(G)^{1/2} \log^2 n).$$

Furthermore, if $\frac{n-\Delta}{\log n} \to \infty$, then as $n \to \infty$, almost every Δ-regular graph G on n vertices has

$$\mathrm{cp}(K_n \backslash G) = \Omega\left(\Delta^2 \left[\frac{\log n}{\log \Delta} - 1 \right]^2 \right).$$

For the purpose of comparison, if $\Delta = n^{1-\varepsilon}$ in Theorem 2, then the upper bound for $\mathrm{cp}(K_n \backslash G)$ is of order $n^{2-(1/2)\varepsilon} \log^2 n$ whereas the lower bound is of order $n^{2-2\varepsilon}$. It would be interesting to determine whether either of these bounds is tight in order of magnitude. We conclude with the following conjecture:

Conjecture 3. Let G be a graph on n vertices with maximum degree Δ. Then $\mathrm{cp}(K_n \backslash G) = \mathrm{O}(\Delta n \log n)$. Furthermore, if $\Delta = \mathrm{o}(n)$, then $\mathrm{cp}(K_n \backslash G) = \mathrm{o}(n^2)$.

2. Complements of forests

To prove Theorem 1, we will restrict our attention to trees and show $\mathrm{cp}(K_n \backslash T) = \mathrm{O}(n \log n)$ for any tree T on n vertices. Theorem 1 follows from this statement, since every forest $F \subset K_n$ is contained in a spanning tree $T \subset K_n$, and

$$\mathrm{cp}(K_n \backslash F) \leqslant \mathrm{cp}(K_n \backslash T) + n - 1.$$

To prove the claim $\mathrm{cp}(K_n \backslash T) = \mathrm{O}(n \log n)$, we make use of the following definition.

Definition 4. A *tree partition* of a graph G is a collection of trees $\{T_1, T_2, \ldots, T_r\}$ of G (not necessarily spanning trees of G) such that every edge of G is in exactly one tree

$$E(G) = \bigcup_{i=1}^{r} E(T_i),$$

and for all $i \neq j$, $|V(T_i) \cap V(T_j)| \leqslant 1$.