European Journal of Combinatorics 61 (2017) 69–84

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

The enumeration of generalized Tamari intervals ☆

Wenjie Fang [a], Louis-François Préville-Ratelle [b]

[a] *IRIF, Université Paris Diderot, France*

[b] *Instituto de Mathemática y Física, Universidad de Talca, Chile*

ARTICLE INFO

Article history:
Received 20 June 2016
Accepted 7 October 2016

ABSTRACT

Let v be a grid path made of north and east steps. The lattice $\textsc{Tam}(v)$, based on all grid paths weakly above v and sharing the same endpoints as v, was introduced by Préville-Ratelle and Viennot (2016) and corresponds to the usual Tamari lattice in the case $v = (NE)^n$. Our main contribution is that the enumeration of intervals in $\textsc{Tam}(v)$, over all v of length n, is given by $\frac{2(3n+3)!}{(n+2)!(2n+3)!}$. This formula was first obtained by Tutte (1963) for the enumeration of non-separable planar maps. Moreover, we give an explicit bijection from these intervals in $\textsc{Tam}(v)$ to non-separable planar maps.

1. Background and main results

The well-known usual Tamari lattice can be defined on Dyck paths or some other combinatorial structures counted by Catalan numbers such as binary trees, and it has many connections with several fields, in particular in algebraic and enumerative combinatorics. In [6], Chapoton showed that the intervals in the Tamari lattice are enumerated by the formula

$$\frac{2}{n(n+1)}\binom{4n+1}{n-1}.$$

☆ An extended abstract of this paper was accepted at the conference on Formal Power Series and Algebraic Combinatorics 2016 (FPSAC 2016), with the title "From generalized Tamari intervals to non-separable planar maps (extended abstract)".

E-mail addresses: wenjie.fang@liafa.univ-paris-diderot.fr (W. Fang), preville-ratelle@inst-mat.utalca.cl (L.-F. Préville-Ratelle).

http://dx.doi.org/10.1016/j.ejc.2016.10.003
0195-6698/

This formula also counts rooted planar 3-connected triangulations. Many results and conjectures about the diagonal coinvariant spaces of the symmetric group (we refer to the books [1,10] for further explanation), also called the Garsia–Haiman spaces, led Bergeron to introduce the m-Tamari lattice for any integer $m \geq 1$. The case $m = 1$ is the usual Tamari lattice. It was conjectured in [2] and proved in [5,4] that the number of intervals and labeled intervals in the m-Tamari lattice of size n is given respectively by the formulas

$$\frac{m+1}{n(mn+1)}\binom{(m+1)^2 n+m}{n-1} \quad \text{and} \quad (m+1)^n(mn+1)^{n-2}.$$

These labeled intervals (resp. unlabeled intervals) are conjectured to be enumerated by the same formulas as the dimensions (resp. dimensions of the alternating component) of the trivariate Garsia–Haiman spaces. These connections motivated the introduction of the lattice $\textsc{Tam}(v)$ in [16] for an arbitrary grid path v as a further generalization. In particular, the Tamari lattice of size n is given by $\textsc{Tam}((NE)^n)$, and more generally the m-Tamari lattice by $\textsc{Tam}((NE^m)^n)$. A precise definition of $\textsc{Tam}(v)$ will be given in Section 2.

The Tamari lattice and its generalizations, while being deeply rooted in algebra, have mysterious enumerative aspects and bijective links yet to be unearthed. For instance, intervals in the Tamari lattice are equi-enumerated with planar triangulations, and a bijection was given by Bernardi and Bonichon in [3]. Similarly, the number of intervals and labeled intervals in the m-Tamari lattice in [5,4] is also given by simple planar-map-like formulas, where a combinatorial explanation is still missing. In this context, similar to the bijection in [3], we also discover a bijection between intervals and maps, contributing to the combinatorial understanding of the Tamari lattice. We should mention that there are also bijective links between Tamari intervals, interval posets and tree flows [8].

In this article, we give an explicit bijection between intervals in $\textsc{Tam}(v)$ and non-separable planar maps, from which we obtain the enumeration formula of these intervals. To describe it, we need two intermediate structures: one called *synchronized interval*, which is a special kind of interval in the usual Tamari lattice; the other called *decorated tree*, basically a kind of rooted tree with labels on their leaves that satisfy certain conditions. The bijection from generalized Tamari intervals to synchronized intervals is implicitly given in [16]. We then show that an exploration process gives a bijection between non-separable planar maps and decorated trees, and we present another bijection between decorated trees and synchronized intervals.

Tutte showed in [18] that non-separable planar maps with $n+2$ edges are counted by $\frac{2(3n+3)!}{(n+2)!(2n+3)!}$. Therefore, we obtain as a consequence of our bijection the following enumeration formula of intervals in $\textsc{Tam}(v)$.

Theorem 1.1. *The total number of intervals in* $\textsc{Tam}(v)$ *over all possible* v *of length* n *is given by*

$$\sum_{v\in(N,E)^n} \mathrm{Int}(\textsc{Tam}(v)) = \frac{2(3n+3)!}{(n+2)!(2n+3)!}. \tag{1}$$

2. From canopy intervals to synchronized intervals

A *grid path* is a (finite) walk on the square grid, starting at $(0, 0)$, consisting of north and east unit steps denoted by N and E respectively. The size of a grid path is the number of steps it contains. For v an arbitrary grid path, let $\textsc{Tam}(v)$ be the set of grid paths that are weakly above v and share the same endpoints as v. The covering relation defined as follows gives $\textsc{Tam}(v)$ a lattice structure. For v_1 a grid path in $\textsc{Tam}(v)$ and p a grid point on v_1, we define the horizontal distance $\mathrm{horiz}_v(p)$ to be the maximum number of east steps that we can take starting from p without crossing v. The left part of Fig. 1 gives an example of a path in $\textsc{Tam}(v)$, with the horizontal distance of each lattice point on v_1. Suppose that p is preceded by a step E and followed by a step N in v_1. Let p' be the first lattice point in v_1 after p such that $\mathrm{horiz}_v(p') = \mathrm{horiz}_v(p)$, and $v_1[p, p']$ the sub-path of v_1 from p to p'. By switching the step E just before p and the sub-path $v_1[p, p']$ in v_1, we obtain another path v_1' in $\textsc{Tam}(v)$. The