European Journal of Combinatorics 60 (2017) 1–9

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

A Ramsey type result for oriented trees

Raphael Yuster

Department of Mathematics, University of Haifa, Haifa 31905, Israel

ARTICLE INFO

Article history:
Received 7 April 2015
Accepted 24 August 2016
Available online 16 September 2016

ABSTRACT

Given positive integers h and k, denote by $r(h, k)$ the smallest integer n such that in any k-coloring of the edges of a tournament on more than n vertices there is a monochromatic copy of every oriented tree on h vertices. We prove that $r(h, k) = (h-1)^k$ for all k sufficiently large ($k = \Theta(h \log h)$ suffices). The bound $(h-1)^k$ is tight. The related parameter $r^*(h, k)$ where some color contains all oriented trees is asymptotically determined. Values of $r(h, 2)$ for some small h are also established.

1. Introduction

An *oriented graph* is a digraph such that for every two distinct vertices u, v at most one of the ordered pairs (u, v) or (v, u) is an edge. Stated otherwise, an oriented graph is obtained by assigning a direction to each edge of an undirected graph. The undirected graph is also called the *underlying graph*. A *tournament* is an oriented graph whose underlying graph is complete. An *oriented tree* is an oriented graph whose underlying graph is a tree.

A seminal theorem, so called the Gallai–Roy Theorem asserts that any oriented graph has a directed path whose order is at least as large as the chromatic number of its underlying graph. This theorem was obtained independently by Gallai [7], Hasse [9], Roy [13], and Vitaver [14]. We note that the Gallai–Roy Theorem generalizes Redei's Theorem [12] that states that any tournament has a Hamilton path.

By observing that in any edge coloring of a complete graph with more than $\prod_{i=1}^{k}(h_i - 1)$ vertices with k colors, there is a color i that induces a graph whose chromatic number is at least h_i, Gyárfás and Lehel [8], Bermond [2], and Chvátal [4] deduced that in any k-coloring of the edges of a tournament on more than $\prod_{i=1}^{k}(h_i - 1)$ vertices, there is a directed path of order h_i, all of whose edges are colored i. They also observed that there is a simple construction showing that the bound $\prod_{i=1}^{k}(h_i - 1)$ is

E-mail address: raphy@math.haifa.ac.il.

http://dx.doi.org/10.1016/j.ejc.2016.08.010
0195-6698/

tight. The diagonal case, where all h_i are equal, is equivalently stated as the following Ramsey-type parameter. Let P_h denote the directed path of order h. Given a positive integer h, let $r(P_h, k)$ be the smallest integer n such that in any k-coloring of the edges of a tournament with more than n vertices, there is a monochromatic P_h. The aforementioned result states that $r(P_h, k) = (h-1)^k$.

A natural question which follows is the value of the corresponding Ramsey number of oriented trees other than the directed path. In particular, what bound guarantees a monochromatic copy of any oriented tree on h vertices? Already the case $k = 1$ is interesting, and, in fact, notoriously difficult. A famous conjecture of Sumner from 1971 states that any tournament on $2h-2$ vertices contains any oriented tree on h vertices (we always assume $h \geq 2$ to avoid the trivial case). If true, then this is best possible since a regular tournament on $2h-3$ vertices has all in-degrees and out-degrees equal to $h-2$. It therefore has no copy of S_h, the out-directed star on h vertices. Sumner's conjecture is still open, though it has recently been established for very large h by Kuhn, Mycroft, and Osthus [11]. The best bound that applies to all h is $3h-3$ proved by El Sahili [6] based on a method of Havet and Thomassé [10].

Let $r(h, k)$ be the smallest integer n such that in any k-coloring of the edges of a tournament with more than n vertices, there is a monochromatic copy of every oriented tree on h vertices. Determining $r(h, 1)$ is thus equivalent to solving Sumner's conjecture. The discussion in the previous paragraphs implies, in particular, that $r(h, k) \geq (h-1)^k$, that $3h-4 \geq r(h, 1) \geq 2h-3$ and that $r(h, 1) = 2h-3$ for all h sufficiently large. Our first main result is the following.

Theorem 1. *Let $h \geq 2$ be a positive integer and let k be a positive integer satisfying $(1 + 1/(h-2))^k > 2(h-2)k + 1$. Then, for every $n > (h-1)^k$, any edge coloring of an n-vertex tournament with k colors contains a monochromatic copy of every oriented tree on h vertices. In particular, $r(h, k) = (h-1)^k$.*

The fact that Theorem 1 requires some lower bound on k in order for the value $(h-1)^k$ to hold is, of course, necessary as shown already for the case $k = 1$. It is thus of some interest to determine, for a given h, the value $f(h)$ which is the smallest k for which $r(h, k) = (h-1)^k$. Theorem 1 shows that $f(h) = O(h \log h)$, but we cannot rule out that $f(h)$ is bounded by a value independent of h. Nevertheless, we certainly have $f(h) \geq 2$ for all $h \geq 3$ as demonstrated by the lower bound in Sumner's conjecture. Furthermore, Sumner's conjecture is known to hold for some small h by computer verification. As usual in Ramsey theory, when the number of colors increases, say even $k = 2$ colors, it is not easy to determine $r(h, 2)$ even for very small h. The fact that $r(3, 2) = 5$ and $r(3, k) = 2^k$ for all $k \geq 3$ is a simple exercise. Hence $f(3) = 3$. Already determining the first non-trivial case $r(4, 2)$ turns out to be somewhat involved, as well as determining $f(h)$ for $h \geq 4$. We show that:

Theorem 2. $r(4, 2) = 12$. *Hence, $f(4) \geq 3$. In fact, $f(h) \geq 3$ for all $h \leq 6$.*

Notice that it is hopeless to use computer verification for $r(4, 2)$ as one needs to check all 2-edge colorings of all (non-isomorphic) tournaments on 13 vertices and it is known that there are more than 2^{45} such tournaments.

One may wonder whether Theorem 1 can be strengthened to show that there is some particular color so that there is a monochromatic copy of every oriented tree with that color. Formally, let $r^*(h, k)$ be the smallest integer n such that in any k-coloring of the edges of a tournament with more than n vertices, some color induces a subgraph that contains all oriented trees on h vertices. Clearly $r^*(h, k) \geq r(h, k)$. However, we show in Proposition 3.1 that $r^*(k, h) \geq (2h-3)(h-1)^{k-1}$ so Theorem 1 does not hold for this stronger parameter, and $r^*(h, k)$ is truly separated from $r(h, k)$. Nevertheless, we can prove that the bound $(h-1)^k$ is asymptotically correct in the sense that the base $h-1$ can be replaced with $h-1+\epsilon$.

Theorem 3. *For every $\epsilon > 0$, an integer $h \geq 2$ and a positive integer k satisfying $(1 + \epsilon/(h-1))^k > 2(h-2)k + 1$ we have $r^*(h, k) \leq (h-1+\epsilon)^k$.*

It is appropriate to mention here the conjecture of Burr [3] that any digraph whose chromatic number is at least $2h-2$ contains every oriented tree on h vertices. It has been proved by Addario-Berry, Havet, Sales, Reed, and Thomassé [1] that chromatic number $h^2/2 - h/2 + 1$ suffices. Hence, if Burr's