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A Ramsey theorem for partial orders with linear extensions

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ABSTRACT

We prove a Ramsey theorem for finite sets equipped with a partial order and a fixed number of linear orders extending the partial order. This is a common generalization of two recent Ramsey theorems due to Sokić. As a bonus, our proof gives new arguments for these two results.

1. The theorem

In recent years, there has been a renewed interest in Structural Ramsey Theory sparked by the discovery in [6] of connections between this area and Topological Dynamics. Papers [2,9] give surveys of these developments. In this context, some attention was directed towards the so-called mixed structures obtained by superimposing a number of simpler structures that are known to be Ramsey; see [9, Section 5.7]. A general Ramsey theorem for such structures was proved in [1] (see also [13]) under the additional assumption that the superimposed structures are independent from each other. The present work contributes a particular structural Ramsey theorem to this area, where the superimposed structures are not independent, but rather are interconnected in a natural way.

In this paper, all orders are strict orders.

For the rest of the paper, we fix a natural number $p > 0$.

By a *structure* we understand a set X equipped with a partial order P and p linear orders $L_0, \ldots, L_{p-1}$ each of which extends P. We write

$$\vec{L}$$

for $(L_0, \ldots, L_{p-1})$ and

$$(X, P, \vec{L})$$

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for the whole structure. A structure is called finite if X is a finite set. Given two structures $\mathcal{X} = (X, P^X, \vec{L}^X)$ and $\mathcal{Y} = (Y, P^Y, \vec{L}^Y)$, a function $f : X \to Y$ is an *embedding* if for all $x_1, x_2 \in X$

$$x_1 P^X x_2 \iff f(x_1) P^Y f(x_2)$$

and, for each $i < p$,

$$x_1 L_i^X x_2 \iff f(x_1) L_i^Y f(x_2).$$

By a *copy* we understand the image of an embedding.

For a natural number $d > 0$, a *d-coloring* is a coloring with d colors.

Theorem 1. *Let $d > 0$, and let $\mathcal{X} = (X, P^X, \vec{L}^X)$ and $\mathcal{Y} = (Y, P^Y, \vec{L}^Y)$ be finite structures. There exists a finite structure $\mathcal{Z} = (Z, P^Z, \vec{L}^Z)$ with the following property: for each d-coloring of all copies of $\mathcal{X}$ in $\mathcal{Z}$, there exists a copy $\mathcal{Y}'$ of $\mathcal{Y}$ in $\mathcal{Z}$ such that all copies of $\mathcal{X}$ in $\mathcal{Y}'$ have the same color.*

The theorem above gives a common generalization of the following two of its known special cases. The first one is the case $p = 1$, that is, the case when structures are equipped with a partial order and a single linear order extending it. This case follows from the results of Nešetřil and Rödl [8] and is explicitly stated in [7]. It was later reproved by Sokić [11, Theorem 7(6)] using results of Paoli, Trotter and Walker [10] and Fouché [4]. The second case is the case of finite sets endowed only with p linear orders. This situation corresponds to $P^X = P^Y = \emptyset$ (when one can obviously make $P^Z = \emptyset$) in Theorem 1. It was proved for $p = 2$ by Böttcher in 2005, see [3], and for arbitrary p by Sokić [12, Theorem 10]. Our proof here is different from the ones employed in the special cases.

In our proofs, we use some ideas from [4,10]. We connect them with the main theorem from [14].

The proof of Theorem 1 is structured as follows. In Section 2, we prove a product Ramsey theorem that is the Ramsey theoretic core of Theorem 1. In Sections 3 and 4, we make explicit certain canonical structures and morphisms important to the proof. Once these structures are properly defined and their natural properties are established, the theorem is proved by appropriately interpreting the objects involved in it and applying the product Ramsey theorem from Section 2. This is done in Section 5. Section 6 has an explanatory character. In it, we make precise the relationship between the product Ramsey theorem and Theorem 1 using general notions introduced [15].

2. A product Ramsey theorem

As promised in Section 1, we prove here a product Ramsey result, Proposition 2, needed in our proof of Theorem 1. We establish it as a consequence of two known Ramsey theorems.

We adopt the notational convention that each natural number is equal to the set of its predecessors, that is,

$$m = \{i : i < m\}.$$

In particular, $0 = \emptyset$. The set m is considered to be linearly ordered with its natural order inherited from $\mathbb{N}$. For a set X and a natural number k,

$$\binom{X}{k}$$

is the family of all k element subsets of X. The set X can itself be a natural number m and then $\binom{m}{k}$ is the family of all k element subsets of m.

Let A, B be two finite linearly ordered sets. A function $r : B \to A$ is a *rigid surjection* if it is a surjection and the images of initial segments of B are initial segments of A, in other words, if for all $a_1, a_2 \in A$, with a_1 preceding a_2 in A, we have that a_1 is first attained by r before a_2 is first attained by r. The language of rigid surjections can be translated into the language of partitions. We formulate all our results in terms of rigid surjections, rather than partitions, as this framework fits the applications better; see Lemma 6 and the proof of Lemma 7(ii). The reader may consult [15] for more information on rigid surjections, partitions, and their equivalence.