# On degree anti-Ramsey numbers 

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## A R T I CLE I N F O

## Article history:

Received 3 October 2015
Accepted 7 September 2016
Available online 2 October 2016


#### Abstract

The degree anti-Ramsey number $A R_{d}(H)$ of a graph $H$ is the smallest integer $k$ for which there exists a graph $G$ with maximum degree at most $k$ such that any proper edge colouring of $G$ yields a rainbow copy of $H$. In this paper we prove a general upper bound on degree anti-Ramsey numbers, determine the precise value of the degree anti-Ramsey number of any forest, and prove an upper bound on the degree anti-Ramsey numbers of cycles of any length which is best possible up to a multiplicative factor of 2 . Our proofs involve a variety of tools, including a classical result of Bollobás concerning cross intersecting families and a topological version of Hall's Theorem due to Aharoni, Berger and Meshulam.


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## 1. Introduction

A copy of a graph $H$ in an edge coloured graph $G$ is called rainbow if all edges of this copy of $H$ have distinct colours. The degree anti-Ramsey number $A R_{d}(H)$ of a graph $H$ is the smallest integer $k$ for which there exists a graph $G$ with maximum degree at most $k$ such that any proper edge colouring of $G$ yields a rainbow copy of $H$. This notion, which is the focus of this paper, was introduced in [2].

Several versions of anti-Ramsey numbers appear in the literature (see, e.g., [9] and the many references therein). The local anti-Ramsey number $A R(H)$ of a graph $H$ is the smallest integer $n$ such that any proper edge colouring of $K_{n}$ yields a rainbow copy of $H$. This graph invariant was studied by various researchers, including Babai [6] and Alon, Lefmann and Rödl [4]. As noted in [2], it is evident that

$$
\begin{equation*}
A R_{d}(H) \leq A R(H)-1 \text { holds for any graph } H . \tag{1}
\end{equation*}
$$

[^0]The size anti-Ramsey number $A R_{s}(H)$ of a graph $H$ is the smallest integer $m$ for which there exists a graph $G$ with $m$ edges such that any proper edge colouring of $G$ yields a rainbow copy of $H$. This graph invariant was introduced by Axenovich, Knauer, Stumpp and Ueckerdt [5] who proved upper and lower bounds on the size anti-Ramsey numbers of paths, cycles, matchings and cliques. In [2], Alon proved that $A R_{d}\left(K_{k}\right)=\Theta\left(k^{3} / \log k\right)$ and used this result to prove that $A R_{s}\left(K_{k}\right)=\Omega\left(k^{6} / \log ^{2} k\right)$, thus settling a problem of Axenovich et al. [5].

It readily follows from (1) that any upper bound on $\operatorname{AR}(H)$ immediately translates to an upper bound on $A R_{d}(H)$. One such upper bound was proved by Alon, Jiang, Miller and Pritikin in [3]. It was proved there that for every graph $H=(V, E)$ with maximum degree $\Delta$,

$$
\begin{equation*}
A R(H) \leq 2 \Delta^{2}|V|+32 \Delta^{4}+4|V| . \tag{2}
\end{equation*}
$$

Our first result is an improvement of (2) which, in particular, is independent of $\Delta$. Instead, it depends on the degeneracy of $H=(V, E)$, i.e., the smallest integer $r$ for which there exists an ordering $v_{1}, \ldots, v_{h}$ of the vertices of $H$ such that $\left|\left\{1 \leq j<i: v_{j} v_{i} \in E\right\}\right| \leq r$ holds for every $1 \leq i \leq h$. Moreover, our proof is elementary and arguably simpler than the probabilistic argument used in [3].

Theorem 1.1. Let $H=(V, E)$ be a graph and let $r$ be its degeneracy. Then

$$
A R(H) \leq r|E|-r+|V|
$$

Note that Theorem 1.1 is indeed an improvement of $(2)$, since if $H=(V, E)$ is graph with maximum degree $\Delta$ and degeneracy $r$, then

$$
r|E|-r+|V| \leq r^{2}|V|+|V| \leq \Delta^{2}|V|+|V| .
$$

Using (1) we obtain the following immediate consequence of Theorem 1.1:
Corollary 1.2. Let $H=(V, E)$ be a graph and let $r$ be its degeneracy. Then

$$
A R_{d}(H) \leq r|E|-r+|V|-1
$$

As observed in [2], it readily follows from Vizing's Theorem [11] that

$$
\begin{equation*}
A R_{d}(H) \geq e(H)-1 \text { holds for any graph } H . \tag{3}
\end{equation*}
$$

It was also observed in [2] that (3) is tight whenever $H$ is a matching with at least 3 edges (it is obvious that $A R_{d}\left(K_{2}\right)=1$ and easy to see that $A R_{d}\left(2 K_{2}\right)=2$ ). Moreover, it was noted in [2] that (3) is almost tight for forests, i.e., $A R_{d}(H) \leq e(H)$ whenever $H$ is a forest. Our next result determines the precise value of the degree anti-Ramsey number of every forest.

Theorem 1.3. Let $F$ be a forest. Then $A R_{d}(F)=e(F)-1$, unless $F$ is a star of any size or a matching with precisely two edges, in which case $A R_{d}(F)=e(F)$.

Finally, we study degree anti-Ramsey numbers of cycles. It readily follows from (3) and Corollary 1.2 that $k-1 \leq A R_{d}\left(C_{k}\right) \leq 3(k-1)$ holds for every $k \geq 3$. Our next result improves the upper bound.

Theorem 1.4. For every $k \geq 3$,

$$
A R_{d}\left(C_{k}\right) \leq \begin{cases}2(k-1) & \text { if } k \text { is even } \\ 2(k+2) & \text { if } k \text { is odd. }\end{cases}
$$

For some small values of $k$ we can prove sharper bounds. It is obvious that $A R_{d}\left(C_{3}\right)=2$ and we can prove that $A R_{d}\left(C_{5}\right) \leq 6$ (this will be discussed at the end of Section 4). Our next result determines the exact value of $A R_{d}\left(C_{4}\right)$.

## Proposition 1.5.

$$
A R_{d}\left(C_{4}\right)=4
$$

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    http://dx.doi.org/10.1016/j.ejc.2016.09.002
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