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# Local colourings and monochromatic partitions in complete bipartite graphs

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ABSTRACT

We show that for any 2-local colouring of the edges of the balanced complete bipartite graph $K_{n,n}$, its vertices can be covered with at most 3 disjoint monochromatic paths. And, we can cover almost all vertices of any complete or balanced complete bipartite $r$-locally coloured graph with $O(r^2)$ disjoint monochromatic cycles. We also determine the 2-local bipartite Ramsey number of a path almost exactly: Every 2-local colouring of the edges of $K_{n,n}$ contains a monochromatic path on $n$ vertices.



## 1. Introduction

The problem of partitioning a graph into few monochromatic paths or cycles, first formulated explicitly in the beginning of the 80s [11], has lately received a fair amount of attention. Its origin lies in Ramsey theory and its subject is complete graphs (later substituted with other types of graphs), whose edges are coloured with $r$ colours. Call such a colouring an $r$-colouring; note that this need not be a proper edge-colouring. The challenge is now to find a small number of disjoint monochromatic paths, which together cover the vertex set of the underlying graph. Or, instead of disjoint monochromatic paths, we might ask for disjoint monochromatic cycles. Here, single vertices and edges count as cycles. Such a cover is called a monochromatic path partition, or a monochromatic cycle partition, respectively. It is not difficult to construct $r$-colourings that do not allow for partitions into less than $r$ paths, or cycles.[1]

At first, the problem was studied mostly for $r = 2$, and the complete graph $K_n$ as the host graph. In this situation, a partition into two disjoint paths always exists [10], regardless of the size of $n$.

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[1] For instance, take vertex sets $V_1, \ldots, V_r$ with $|V_i| = 2^i$, and for $i \leq j$ give all $V_i - V_j$ edges colour $i$.

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Moreover, we can require these paths to have different colours. An extension of this fact, namely that every 2-colouring of $K_n$ has a partition into two monochromatic cycles of different colours was conjectured by Lehel, and verified by Bessy and Thomassé [3], after preliminary work for large $n$ [1,22].

A generalisation of these two results for other values of $r$, i.e. that any $r$-coloured $K_n$ can be partitioned into $r$ monochromatic paths, or into $r$ monochromatic cycles, was conjectured by Gyárfás [12] and by Erdős, Gyárfás and Pyber [7], respectively. The conjecture for cycles was recently disproved by Pokrovskiy [24]. He gave counterexamples for all $r \geq 3$, but he also showed that the conjecture for paths is true for $r = 3$. Gyárfás, Ruszinkó, Sárközy and Szemerédi [16] showed that any $r$-coloured $K_n$ can be partitioned into $O(r \log r)$ monochromatic cycles, improving an earlier bound from [7].

Monochromatic path/cycle partitions have also been studied for bipartite graphs, mainly for $r = 2$. A 2-colouring of $K_{n,n}$ is called a split colouring if there is a colour-preserving homomorphism from the edge-coloured $K_{n,n}$ to a properly edge-coloured $K_{2,2}$. Note that any split colouring allows for a partition into three paths, but not always into two. However, split colourings are the only 'problematic' colourings, as the following result shows.

**Theorem 1.1** (*Pokrovskiy [24]*)**.** *Let the edges of $K_{n,n}$ be coloured with 2 colours; then $K_{n,n}$ can be partitioned into two paths of distinct colours or the colouring is split.*

Split colourings can be generalised to more colours [24]. This gives a lower bound of $2r - 1$ on the path/cycle partition number for $K_{n,n}$. For $r = 3$, this bound is asymptotically correct [20]. For an upper bound, Peng, Rödl and Ruciński [23] showed that any $r$-coloured $K_{n,n}$ can be partitioned into $O(r^2 \log r)$ monochromatic cycles, improving a result of Haxell [19]. We improve this bound to $O(r^2)$.

**Theorem 1.2.** *For every $r \geq 1$ there is an $n_0$ such that for $n \geq n_0$, for any $r$-locally coloured $K_{n,n}$, we need at most $4r^2$ disjoint monochromatic cycles to cover all its vertices.*

Theorem 1.2 follows immediately from Theorem 1.3(b). Let us mention that the monochromatic cycle partition problem has also been studied for multipartite graphs [29], and for arbitrary graphs [2,26], or replacing paths or cycles with other graphs [9,27,28].

Our main focus in this paper is on monochromatic cycle partitions for *local colourings* (Theorem 1.2 being only a side-product of our local colouring results). Local colourings are a natural way to generalise $r$-colourings. A colouring is $r$-local if no vertex is adjacent to more than $r$ edges of distinct colours. Local colourings have appeared mostly in the context of Ramsey theory [4,5,14,15,25,30–32].

With respect to monochromatic path or cycle partitions, Conlon and Stein [6] recently generalised some of the above mentioned results to $r$-local colourings. They show that for any $r$-local colouring of $K_n$, there is a partition into $O(r^2 \log r)$ monochromatic cycles, and, if $r = 2$, then two cycles suffice. In this paper we improve their general bound for complete graphs, and give the first bound for monochromatic cycle partitions in bipartite graphs. In both cases, $O(r^2)$ cycles suffice.

**Theorem 1.3.** *For every $r \geq 1$ there is an $n_0$ such that for $n \geq n_0$ the following holds.*

(a) *If $K_n$ is $r$-locally coloured, then its vertices can be covered with at most $2r^2$ disjoint monochromatic cycles.*
(b) *If $K_{n,n}$ is $r$-locally coloured, then its vertices can be covered with at most $4r^2$ disjoint monochromatic cycles.*

We do not believe our results are best possible, but suspect that in both cases ($K_n$ and $K_{n,n}$), the number of cycles needed should be linear in $r$.

**Conjecture 1.4.** *There is a $c$ such that for every $r$, every $r$-local colouring of $K_n$ admits a covering with $cr$ disjoint cycles. The same should hold replacing $K_n$ with $K_{n,n}$.*

Our second result is a generalisation of Theorem 1.1 to local colourings:

**Theorem 1.5.** *Let the edges of $K_{n,n}$ be coloured 2-locally. Then $K_{n,n}$ can be partitioned into 3 or less monochromatic paths.*