# On the zero-one $k$-law extensions 

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#### Abstract

The presented paper is devoted to the asymptotical behavior of first-order properties of the Erdős-Rényi random graph. In previous works the zero-one $k$-law was proved. This law describes asymptotical behavior of first-order properties which are expressed by formulae with a quantifier depth bounded by $k$. The random graph $G\left(N, N^{-\alpha}\right)$ obeys the law if $\alpha \in(0,1 /(k-2))$. In this work we find new values of $\alpha$, which are close to 1 , such that $G\left(N, N^{-\alpha}\right)$ obeys the zero-one $k$-law and, therefore, extend the previous result.


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## 1. Introduction

In this section we introduce the history of the problem, give definitions and state the main result.
Let $N \in \mathbb{N}, 0 \leq p \leq 1$. Consider the set $\Omega_{N}=\left\{G=\left(V_{N}, E\right)\right\}$ of all undirected graphs without loops and multiple edges with the set of vertices $V_{N}=\{1,2, \ldots, N\}$. The Erdös-Rényi random graph (see [1,3,7,10]) is a random element $G(N, p)$ of $\Omega_{N}$ with a distribution $\mathrm{P}_{N, p}$ on $\mathcal{F}_{N}=2^{\Omega_{N}}$ defined as follows:

$$
\mathrm{P}_{N, p}(G)=p^{|E|}(1-p)^{C_{N}^{2}-|E|}
$$

The random graph obeys the zero-one law with a class of properties $\mathcal{C}$ if for each property $L \in \mathcal{C}$ either its probability tends to 0 or tends to 1 .

The class of the first-order properties is the most studied class in this area. Such properties are expressed by first-order formulae (see [2,5]). These formulae are built of the predicate symbols $\sim,=$; the logical connectivities $\neg, \Rightarrow, \Leftrightarrow, \vee, \wedge$; the variables $x, y, x_{1} \ldots$; the quantifiers $\forall, \exists$. Symbols $x, y, x_{1} \ldots$ express vertices of a graph. The relation symbol $\sim$ expresses the property of two vertices

[^0]to be adjacent. The symbol $=$ expresses the property of two vertices being coincident. We denote by $\mathcal{P}$ a class of functions $p=p(N)$ such that the random graph $G(N, p)$ obeys the zero-one law with the class $\mathscr{L}$ of all the first-order properties. In 1969 by Y.V. Glebskii, D.I. Kogan, M.I. Liagonkii and V.A. Talanov in [9] (and independently in 1976 R. Fagin in [8]) proved that if
$$
\forall \alpha>0 \quad N^{\alpha} \min \{p, 1-p\} \rightarrow \infty, N \rightarrow \infty,
$$
then $p \in \mathcal{P}$. Moreover, in 1988 S. Shelah and J.H. Spencer (see [13,15]) extended the class of functions $p(N)$ "that obeys the zero-one law". They proved that the functions $p=N^{-\alpha}, \alpha \in \mathbb{R} \backslash \mathbb{Q}, \alpha \in(0,1)$, are in $\mathcal{P}$. Indeed, $p=1-N^{-\alpha} \in \mathscr{P}$ when $\alpha \in \mathbb{R} \backslash \mathbb{Q}, \alpha \in(0,1)$.

If $\alpha$ is rational, $0<\alpha \leq 1$ and $p=N^{-\alpha}$, then $G(N, p)$ does not obey the zero-one law (see [1]).
Denote by $\mathcal{L}^{\infty}, \mathscr{L}^{\infty} \supset \mathcal{L}$ the class of all properties expressed by formulae containing infinite (or finite) number of conjunctions and disjunctions. The class $\mathscr{L}_{k}^{\infty}, \mathscr{L}_{k}^{\infty} \subset \mathcal{L}^{\infty}$ containing all properties expressed by formulae with a quantifier depth bounded by $k$ in the frame of the zero-one laws was considered by M. McArthur in 1997 [11] (the quantifier depth of a sentence is the number of nested quantifiers). M. McArthur obtained the zero-one laws with the class $\mathcal{L}_{k}^{\infty}$ for the random graph $G\left(N, N^{-\alpha}\right)$ with some rational $\alpha$ from ( 0,1$]$.

Finally, consider the class $\mathscr{L}_{k}=\mathscr{L} \cap \mathcal{L}_{k}^{\infty}$. In 2010 (see [18,16]) we proved that if $k \geq 3$, $\alpha \in(0,1 /(k-2))$ the random graph $G\left(N, N^{-\alpha}\right)$ obeys the zero-one law with the class $\mathscr{L}_{k}$. We also proved that when $\alpha=1 /(k-2)$ the random graph $G\left(N, N^{-\alpha}\right)$ does not obey the zero-one law with this class. In [19] we proved that for any property $L \in \mathscr{L}_{k}$ probability $\mathrm{P}_{N, N^{-1 /(k-2)}}(L)$ converges.

Let us state the main results of the paper. In what follows we consider the only case $k>3$. If $k=3$ (or $k<3$ ) the random graph $G\left(N, N^{-\alpha}\right)$ obeys the zero-one $k$-law for all $\alpha \in(0,1)$ (see, for example, [19]).

Theorem 1. Let $k>3$ be an arbitrary natural number. Moreover let $Q$ be the set of positive rational numbers with the numerator less than or equal to $2^{k-1}$. The random graph $G\left(N, N^{-\alpha}\right)$ obeys the zero-one $k$-law, if $\alpha=1-\frac{1}{2^{k-1}+\beta}, \beta \in(0, \infty) \backslash Q$.

So, we consider the interval $\left(1-2^{1-k}, 1\right)$ and find a set of rational numbers $\alpha$ from the interval such that the random graph obeys the zero-one $k$-law. As any number from $\left(1-2^{1-k}, 1\right)$ can be expressed as $1-\frac{1}{2^{k-1}+\beta}$, the law holds for any $\alpha$ from

$$
\begin{aligned}
\left(1-\frac{1}{2^{k}}, 1\right) & \cup\left(1-\frac{1}{2^{k}-1}, 1-\frac{1}{2^{k}}\right) \bigcup \ldots \bigcup\left(1-\frac{1}{2^{k-1}+2^{k-2}}, 1-\frac{1}{2^{k-1}+2^{k-2}+1}\right) \\
& \cup\left(1-\frac{1}{2^{k-1}+\frac{2^{k-1}-1}{2}}, 1-\frac{1}{2^{k-1}+2^{k-2}}\right) \bigcup \cdots \\
& \cup\left(1-\frac{1}{2^{k-1}+\frac{2^{k-1}}{3}}, 1-\frac{1}{2^{k-1}+\frac{2^{k-1}-\left\lfloor\frac{2^{k-1}}{3}\right\rfloor}{2}}\right) \bigcup \cdots,
\end{aligned}
$$

where $\lfloor\cdot\rfloor,\lceil\cdot\rceil$ are the floor function and the ceiling function respectively. The lengths of the intervals decrease and tend to 0 as the endpoints of the intervals tend to $1-2^{1-k}$. Moreover, we prove that for the endpoints of several intervals the zero-one $k$-law does not hold (see Theorem 2).

The scheme of Theorem 1 proof is the following. In Section 3 we describe the Ehrenfeucht game and formulate a theorem proved by A. Ehrenfeucht which associates the zero-one laws and the existence of winning strategy of the second player in the Ehrenfeucht game. We continue the section with the description of a strategy of the second player. In Section 4 we prove that this strategy is the winning one with probability which tends to 1 . In Section 2 we build additional constructions. Using their properties we prove that the second player is able to exploit the strategy asymptotically almost surely.

We disprove the zero-one $k$-law in the case $\alpha=1-\frac{1}{2^{k-1}+\beta}$ for some values of $\beta$ from the set $\mathcal{Q}$, which is defined in Theorem 1.

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