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Automorphism groups of supergraphs of the power graph of a finite group

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ABSTRACT

For a finite group G, the **power graph** $\mathcal{P}(G)$ is a graph with the vertex set G, in which two distinct elements are adjacent if one is a power of the other. Feng, Ma and Wang (Feng et al., 2016) described the full automorphism group of $\mathcal{P}(G)$. In this paper, we study automorphism groups of the main supergraphs and cyclic graphs, which are supergraphs of $\mathcal{P}(G)$. It is proved that the automorphism group of these graphs can be written as a combination of Cartesian and wreath products of some symmetric groups. The full automorphism groups of these graphs of certain finite groups are also calculated.

1. Introduction

By an **undirected graph** Γ, we mean a pair $(V(\Gamma), E(\Gamma))$ in which $V(\Gamma)$ is a non-empty set and $E(\Gamma)$ is a subset of all unordered pairs of distinct elements of $V(\Gamma)$. If the elements of $E(\Gamma)$ are ordered pairs, then the graph Γ will be directed. The notations $V(\Gamma)$ and $E(\Gamma)$ stand for the set of all vertices and edges of Γ, respectively. The cardinality of $V(\Gamma)$ is called the order of Γ. If $e \in E(\Gamma)$ has end vertices u and v, then we say that u and v are adjacent and this edge is denoted by uv. If $u \in V(\Gamma)$, then $N_\Gamma(u)$ is the set of neighbors of u in Γ, that is, $N_\Gamma(u) = \{v \in V(\Gamma) \mid uv \in E(\Gamma)\}$. The cardinality of $N_\Gamma(u)$ is said to be the degree of u and denoted by $deg(u)$. The graph Γ is called p-regular when every vertex has the same degree equal to p.

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Sabidussi [22, p. 396], has defined the A-join of a set of graphs $\{G_a\}_{a\in A}$ as the graph H with vertex and edge sets

$$V(H) = \{(x, y) \mid x \in V(A) \,\&\, y \in V(G_x)\},$$
$$E(H) = \{(x, y)(x', y') \mid xx' \in E(A) \text{ or else } x = x' \,\&\, yy' \in E(G_x)\}.$$

This graph is obtained by replacing each vertex $a \in V(A)$ by the graph G_a and inserting either all or none of the possible edges between vertices of G_a and G_b depending on whether or not a and b are joined by an edge in A. If A is labeled and has p points, then the A-join of $H_1, H_2, \ldots, H_p$ is denoted by $A[H_1, H_2, \ldots, H_p]$.

A graph Γ is said to be a **subgraph** of another graph Δ (or Δ is a **supergraph** of Γ), if $V(\Gamma) \subseteq V(\Delta)$ and $E(\Gamma) \subseteq E(\Delta)$. The length of a minimal path connecting vertices x and y in a simple graph Γ is called **topological distance** between x and y and the maximum of such topological distances is the **diameter** of Γ. The topological distance between x and y and diameter of Γ are denoted by $d_\Gamma(x, y)$ and $diam(\Gamma)$, respectively.

There are several kinds of simple graphs associated to a finite group G in which the oldest one is famous **Cayley graph**. In this paper, we are interesting in the well-known **power graph** $\mathcal{P}(G)$ and its **main supergraph** $\mathcal{S}(G)$ with the same vertex set G. Two elements $x, y \in G$ are adjacent in the power graph if and only if one is a power of the other and they are joined to each other in $\mathcal{S}(G)$ if and only if $o(x)|o(y)$ or $o(y)|o(x)$. The **proper power graph** $\mathcal{P}^*(G)$ and its **proper main supergraph** $\mathcal{S}^*(G)$ are defined as graphs constructed from $\mathcal{P}(G)$ and $\mathcal{S}(G)$ by removing identity element of G, respectively [6,12]. Since every non-identity element is adjacent with identity in $\mathcal{P}(G)$ and $\mathcal{S}(G)$, these graphs are connected with diameter ≤ 2.

The **cyclic graph** of a group G, Γ_G, was introduced by Abdollahi and Mohammadi Hassanabadi [3]. In [4], the same authors characterized groups whose non-cyclic graphs have clique numbers at most 4. In the original definition of this graph it is assumed that $V(\Gamma_G) = G \setminus Cyc(G)$, where $Cyc(G) = \{x \in G \mid \langle x, y\rangle \textit{ is cyclic}; \forall y \in G\}$. Ma et al. [17], considered the whole G as the vertex set of this graph and two vertices x and y are adjacent in Γ_G if and only if $\langle x, y\rangle$ is a cyclic subgroup of G. They proved that the clique number of this graph is equal to $o(a)$, where a has maximum order in G. In a recent paper, Aalipour et al. [1] considered the name **enhanced power graph** for Γ_G, when the vertex set is the whole of G.

The aim of this paper is studying the following question:

Question 1.1. Which finite groups can be presented as automorphism group of $\mathcal{P}(G)$, $\mathcal{S}(G)$ or Γ_G?

The study of power graph was started by publishing the seminal paper of Kelarev and Quinn [13]. In this paper, the authors considered the directed power graph of groups and semigroups into account. The main result of the mentioned paper gives a very technical description of the power graph structure of finite abelian groups. The same authors [16] studied also the power graph of the multiplicative subsemigroup of the ring of $n \times n$ matrices over a skew-field. The interested readers can be consulted [2,15,14] for more information about the power graphs of semigroups.

Chakrabarty et al. [9] introduced the undirected power graph of a finite group and proved that this graph is complete if and only if G is a cyclic p-group, for a prime number p. Then Cameron and Ghosh [8] proved that two abelian groups with isomorphic power graphs must be isomorphic and conjectured that two finite groups with isomorphic power graphs have the same number of elements of each order. Cameron [7] responded affirmatively this conjecture. We refer the interested readers to [5,18–20], for more information on the power graph and the automorphism group of certain finite groups.

Throughout this paper we refer to [21,23] for our group theory and graph theoretical concepts and notations, respectively. The notation S_n stands for the symmetric group on $\{1, 2, \ldots, n\}$, φ is the Euler's totient function and $H \wr K$ denotes the wreath product of the groups H over K.

2. Main results

Feng et al. [11], computed the full automorphism group of the power graph of a finite group. In this section, we apply the main result of the mentioned paper to obtain a complete description of the full