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Decomposing graphs into a constant number of locally irregular subgraphs

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ABSTRACT

A graph is *locally irregular* if no two adjacent vertices have the same degree. The *irregular chromatic index* $\chi'_{\mathrm{irr}}(G)$ of a graph G is the smallest number of locally irregular subgraphs needed to edge-decompose G. Not all graphs have such a decomposition, but Baudon, Bensmail, Przybyło, and Woźniak conjectured that if G can be decomposed into locally irregular subgraphs, then $\chi'_{\mathrm{irr}}(G) \leq 3$. In support of this conjecture, Przybyło showed that $\chi'_{\mathrm{irr}}(G) \leq 3$ holds whenever G has minimum degree at least 10^{10}.

Here we prove that every bipartite graph G which is not an odd length path satisfies $\chi'_{\mathrm{irr}}(G) \leq 10$. This is the first general constant upper bound on the irregular chromatic index of bipartite graphs. Combining this result with Przybyło's result, we show that $\chi'_{\mathrm{irr}}(G) \leq 328$ for every graph G which admits a decomposition into locally irregular subgraphs. Finally, we show that $\chi'_{\mathrm{irr}}(G) \leq 2$ for every 16-edge-connected bipartite graph G.

1. Introduction

A graph G is *locally irregular* if any two of its adjacent vertices have distinct degrees. An edge-weighting of G is called *neighbour-sum-distinguishing*, if for every two adjacent vertices of G the sums of their incident weights are distinct. The least number k for which G admits a neighbour-sum-distinguishing edge-weighting using weights $1, 2, \ldots, k$ is denoted $\chi'_{\Sigma}(G)$.

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Karoński, Łuczak, and Thomason [5] made the following conjecture.

Conjecture 1.1 (*1-2-3 Conjecture [5]*). *For every graph G with no component isomorphic to K_2, we have $\chi'_\Sigma(G) \le 3$.*

This conjecture is equivalent to stating that a graph can be made locally irregular by replacing some of its edges by two or three parallel edges. Although the 1-2-3 Conjecture has received considerable attention in the last decade, it is still an open question. The best result so far was shown by Kalkowski, Karoński, and Pfender [4] who proved $\chi'_\Sigma(G) \le 5$ whenever G has no component isomorphic to K_2. For more details, we refer the reader to the survey by Seamone [8] on the 1-2-3 Conjecture and related problems.

If a graph G is regular, then G admits a neighbour-sum-distinguishing 2-edge-weighting if and only if G can be edge-decomposed into two locally irregular subgraphs. Motivated by this connection, Baudon, Bensmail, Przybyło, and Woźniak [1] asked the more general question when a graph can be edge-decomposed into locally irregular subgraphs, and how many locally irregular subgraphs are needed. From now on, all graphs we consider are simple and finite. A decomposition into locally irregular subgraphs can be regarded as an improper edge-colouring where each colour class induces a locally irregular graph. We call such an edge-colouring *locally irregular*. If G admits a locally irregular edge-colouring, then we call G *decomposable*. For every decomposable graph G, we define the *irregular chromatic index of* G, denoted by $\chi'_{\mathrm{irr}}(G)$, as the least number of colours in a locally irregular edge-colouring of G. If G is not decomposable, then $\chi'_{\mathrm{irr}}(G)$ is not defined and we call G *exceptional*. The following conjecture has a similar flavour to the 1-2-3 Conjecture.

Conjecture 1.2 (*[1]*). *For every decomposable graph G, we have $\chi'_{\mathrm{irr}}(G) \le 3$.*

Every connected graph of even size can be decomposed into paths of length 2 and is thus decomposable. Hence, all exceptional graphs have odd size and a complete characterisation of exceptional graphs was given by Baudon, Bensmail, Przybyło, and Woźniak [1]. To state this characterisation, we first need to define a family $\mathcal{T}$ of graphs. The definition is recursive:

- The triangle K_3 belongs to $\mathcal{T}$.
- Every other graph in $\mathcal{T}$ can be constructed by (1) taking an auxiliary graph F being either a path of even length or a path of odd length with a triangle glued to one of its ends, then (2) choosing a graph $G \in \mathcal{T}$ containing a triangle with at least one vertex, say v, of degree 2 in G, and finally (3) identifying v with a vertex of degree 1 of F.

In other words, the graphs in $\mathcal{T}$ are obtained by connecting a collection of triangles in a tree-like fashion, using paths with certain lengths, depending on what elements these paths connect. Let us point out that all graphs in $\mathcal{T}$ have maximum degree 3, have odd size, and all of their cycles are triangles.

Theorem 1.3 (*[1]*). *A connected graph is exceptional, if and only if it is* (1) *a path of odd length,* (2) *a cycle of odd length, or* (3) *a member of* $\mathcal{T}$.

The number 3 in Conjecture 1.2 cannot be decreased to 2, since $\chi'_{\mathrm{irr}}(G) = 3$ if G is a complete graph or a cycle with length congruent to 2 modulo 4. Baudon, Bensmail, Przybyło, and Woźniak [1] verified Conjecture 1.2 for several classes of graphs such as trees, complete graphs, and regular graphs with degree at least 10^7. Baudon, Bensmail, and Sopena [2] showed that determining the irregular chromatic index of a graph is NP-complete in general, and that, although infinitely many trees have irregular chromatic index 3, the same problem for trees can be solved in linear time. More recently, Przybyło [7] gave further evidence for Conjecture 1.2 by verifying it for graphs of large minimum degree.

Theorem 1.4 (*[7]*). *If a graph G has minimum degree at least 10^{10}, then $\chi'_{\mathrm{irr}}(G) \le 3$.*

Despite this result, Conjecture 1.2 is still wide open, even in much weaker forms. Until now it was not known whether there exists a constant c such that $\chi'_{\mathrm{irr}}(G) \le c$ holds for every decomposable graph G. This was also an open problem when restricted to bipartite graphs, see [1–3,7].