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Topological cycle matroids of infinite graphs

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ABSTRACT

We prove that the topological cycles of an arbitrary infinite graph together with its topological ends form a matroid. This matroid is, in general, neither finitary nor cofinitary.

1. Introduction

Many theorems about finite graphs and their cycles do not extend to infinite graphs and their finite cycles. However, many such theorems do extend to locally finite graphs together with their topological cycles, see for example [7,17,18,21,15] for a survey. These *topological cycles* are homoeomorphic images of the unit circle in the topological space obtained from the graph by adding certain points at infinity called ends.

Bruhn and Diestel gave an explanation why many of these theorems extended: the topological cycles of a locally finite graph form a matroid [8]. This matroidal point of view allowed for new proof techniques and abstracting the topological properties of the topological cycles often led to simpler proofs. For non-locally finite graphs various notions of end boundaries have been suggested [15], each of which gives rise to its own notion of topological cycles.

To compare these end boundaries we will not refer directly to topology but instead compare the matroids they induce. However for some of these notions, the matroids have finite circuits which are not finite cycles of the graph. A consequence of this is that there are non-isomorphic (3-connected) graphs inducing isomorphic matroids. For others we even do not always get a matroid.

Here we show that the topological end boundary, which had not been considered for this purpose before, lacks these defects. More precisely:

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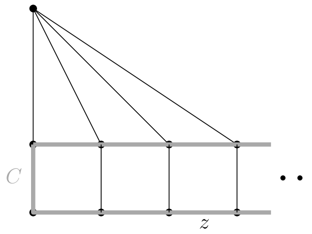

Fig. 1. The dominated ladder is obtained from the one ended ladder by adding a vertex that is adjacent to every vertex on the upper side of the ladder. The topological cycles of VTOP of the dominated ladder do not induce a matroid as they violate the elimination axiom (C3): We cannot eliminate all the triangles from the grey cycle C.

Theorem 1.1. *For any graph G, the topological cycles of G together with its topological ends form a matroid.*

Moreover, for non-isomorphic 3*-connected graphs, these matroids are non-isomorphic.*

Furthermore, all matroids that arise as cycle matroids for one of the other boundaries are minors of these cycle matroids. For the one boundary, where the topological cycles do not induce a matroid for all graphs, Theorem 1.1 implies a characterisation when they do. The various notions of boundary and the corresponding characterisations are compared in Section 1.1.

The question whether the topological cycles in the graph G together with the boundary B induce a matroid is closely related to the question whether G has a spanning tree whose ends are equal to B. Indeed, any such spanning tree is an example of a base in the topological cycle matroid.

In our proof we use a result of [10] which ensures the existence of such spanning trees for the topological ends. We then combine this with the theory of trees of matroids [5].

1.1. Comparing the end boundaries

Bruhn and Diestel showed that the dual of the finite-bond matroid of a graph G is given by the topological cycles of G together with its edge ends [8]. However, after deleting parallel edges, any component of such a matroid is countable.

Hence in order to construct matroids that are nontrivially uncountable, we have to consider topological cycles of different topological spaces. One such space is VTOP, which is obtained from the graph by adding the vertex ends. In Fig. 1, we depicted a graph whose topological cycles in VTOP do not induce a matroid.

The reason why this example works is that the topological cycle C goes through a dominated vertex end. Here a vertex v *dominates* a vertex end ω if there is an infinite v-fan to some ray belonging to ω.

One way to 'repair' VTOP is to identify each vertex ends with the vertices dominating it. The resulting space is called ITOP. A consequence of Theorem 1.1 is the following.

Corollary 1.2. *For any graph, the topological cycles of* ITOP *form a matroid.*

The matroids we get from Corollary 1.2 are more complicated than the ones for the edge ends in the sense that they are not always cofinitary. However, there are still non-isomorphic 3-connected graphs whose ITOP-matroids are isomorphic.

Another way to 'repair' VTOP is to delete the dominated vertex ends. Diestel and Kühn [16] showed that the remaining vertex ends are given by the topological ends, and in this case the topological cycles induce a matroid by Theorem 1.1.

In 1969, Higgs proved that the set of finite cycles and double rays of a graph G is the set of circuits of a matroid if and only if G does not have a subdivision of the Bean-graph [20]. Using Theorem 1.1, we get a result for the topological cycles of VTOP in the same spirit.

Corollary 1.3. *The topological cycles of* VTOP *induce a matroid if and only if G does not have a subdivision of the dominated ladder, which is depicted in Fig. 1.*