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Kempe equivalence of colourings of cubic graphs

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ABSTRACT

Given a graph $G = (V, E)$ and a proper vertex colouring of G, a Kempe chain is a subset of V that induces a maximal connected subgraph of G in which every vertex has one of two colours. To make a Kempe change is to obtain one colouring from another by exchanging the colours of vertices in a Kempe chain. Two colourings are Kempe equivalent if each can be obtained from the other by a series of Kempe changes. A conjecture of Mohar asserts that, for $k \geq 3$, all k-colourings of connected k-regular graphs that are not complete are Kempe equivalent. We address the case $k = 3$ by showing that all 3-colourings of a connected cubic graph G are Kempe equivalent unless G is the complete graph K_4 or the triangular prism.

1. Introduction

Let $G = (V, E)$ denote a simple undirected graph and let k be a positive integer. A *k-colouring* of G is a mapping $\phi : V \to \{1, \dots, k\}$ such that $\phi(u) \neq \phi(v)$ if $uv \in E$. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest k such that G has a k-colouring.

If a and b are distinct colours of a colouring α, then $G_\alpha(a, b)$ denotes the subgraph of G induced by vertices with colour a or b. An (a, b)*-component under* α of G is a connected component of $G_\alpha(a, b)$ and is known as a *Kempe chain* (we will omit the reference to α when it is unneeded). A *Kempe change* is the operation of interchanging the colours of some (a, b)-component of G. Let $C_k(G)$ be the set of all k-colourings of G. Two colourings $\alpha, \beta \in C_k(G)$ are *Kempe equivalent*, denoted by $\alpha \sim_k \beta$, if each

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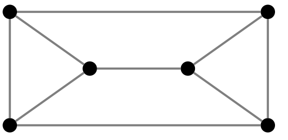

Fig. 1. The 3-prism.

can be obtained from the other by a series of Kempe changes. The equivalence classes $C_k(G)/\sim_k$ are called *Kempe classes*.

Kempe changes were first introduced by Kempe in his well-known failed attempt at proving the Four-Colour Theorem. The Kempe change method has proved to be a powerful tool with applications to several areas such as timetables [16], theoretical physics [21,22], and Markov chains [20]. The reader is referred to [15,17] for further details. From a theoretical viewpoint, Kempe equivalence was first addressed by Fisk [11] who proved that all 4-colourings of an Eulerian triangulation of the plane are Kempe equivalent. This result was later extended by Meyniel [13] who showed that all 5-colourings of a planar graph are Kempe equivalent, and by Mohar [15] who proved that all k-colourings, $k > \chi(G)$, of a planar graph G are Kempe equivalent. Las Vergnas and Meyniel [19] extended Meyniel's result by proving that all 5-colourings of a K_5-minor free graph are Kempe equivalent. Bertschi [2] showed that all k-colourings of a perfectly contractile graph are Kempe equivalent, and, by further showing that any Meyniel graph is perfectly contractile, answered in the affirmative a conjecture of Meyniel [14]. We note that Kempe equivalence with respect to edge-colourings has also been investigated [15,12,1].

Here we are concerned with a conjecture of Mohar [15] on connected *k-regular* graphs, that is, graphs in which every vertex has degree k for some $k \geq 0$. Note that, for every connected 2-regular graph G that is not an odd cycle, it holds that $C_2(G)$ is a Kempe class. Mohar conjectured the following (where K_{k+1} is the complete graph on $k+1$ vertices).

Conjecture 1 *([15]). Let $k \geq 3$. If G is a connected k-regular graph that is not K_{k+1}, then $C_k(G)$ is a Kempe class.*

Notice that if $G = K_{k+1}$, then $C_k(G)$ forms an empty Kempe class; so the condition in Conjecture 1 is not necessary but it is neater to exclude this case. Notice also that if $G \neq K_{k+1}$, then $C_k(G)$ is not empty by Brooks' Theorem [7], which states that a graph with maximum degree k has a k-colouring unless it is an odd cycle or a complete graph.

We address Conjecture 1 for the case $k = 3$. For this case the conjecture is known to be false. A counter-example is the 3-prism displayed in Fig. 1. The fact that some 3-colourings of the 3-prism are not Kempe equivalent was already observed by van den Heuvel [18]. Our contribution is that the 3-prism is the *only* counter-example for the case $k = 3$, that is, we completely settle the case $k = 3$ by proving the following result for 3-regular graphs also known as *cubic* graphs.

Theorem 1. *If G is a connected cubic graph that is neither K_4 nor the 3-prism, then $C_3(G)$ is a Kempe class.*

We give the proof of our result in the next section. Let us note an immediate corollary of our result. First we need a definition and a lemma. Let d be a positive integer. A graph G is *d-degenerate* if every subgraph of G has a vertex with degree at most d.

Lemma 1 *([19,15]). Let d and k be integers, $d \geq 0$, $k \geq d+1$. If G is a d-degenerate graph, then $C_k(G)$ is a Kempe class.*

Corollary 1. *Let G be a connected graph with maximum degree at most 3. Then $C_3(G)$ is a Kempe class unless G is K_4 or the 3-prism.*

Proof. A connected graph with maximum degree 3 is either 3-regular or 2-degenerate (this follows easily from the definition of degenerate, but see also, for example, [10] for a discussion). The corollary follows from Theorem 1 and Lemma 1. □