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On the classification of Stanley sequences

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ABSTRACT

An integer sequence is said to be *3-free* if no three elements form an arithmetic progression. A *Stanley sequence* $\{a_n\}$ is a 3-free sequence constructed by the greedy algorithm. Namely, given initial terms $a_0 < a_1 < \cdots < a_k$, each subsequent term $a_n > a_{n-1}$ is chosen to be the smallest such that the 3-free condition is not violated. Odlyzko and Stanley conjectured that Stanley sequences divide into two classes based on asymptotic growth: Type 1 sequences satisfy $a_n = \Theta(n^{\log_2 3})$ and appear well-structured, while Type 2 sequences satisfy $a_n = \Theta(n^2/\log n)$ and appear disorderly. In this paper, we define the notion of *regularity*, which is based on local structure and implies Type 1 asymptotic growth. We conjecture that the reverse implication holds. We construct many classes of regular Stanley sequences, which include all known Type 1 sequences as special cases. We show how two regular sequences may be combined into another regular sequence, and how parts of a Stanley sequence may be translated while preserving regularity. Finally, we demonstrate the surprising fact that certain Stanley sequences possess proper subsets that are also Stanley sequences.

1. Introduction

A set of nonnegative integers is *3-free* if no three elements form an arithmetic progression. Given a 3-free set A with elements $a_0 < a_1 < \cdots < a_k$, we define the *Stanley sequence* $S(A) = \{a_n\}$ according to the greedy algorithm, as follows: Assuming a_{n-1} has been defined, let a_n be the smallest integer greater than a_{n-1} such that $\{a_0, \ldots, a_n\}$ is 3-free. For convenience, we shall often write $S(a_0, a_1, \ldots, a_k)$ for $S(\{a_0, a_1, \ldots, a_k\})$.

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The simplest Stanley sequence is $S(0) = 0, 1, 3, 4, 9, 10, 12, 13, 27, \ldots,$ the elements of which are exactly those integers with no 2's in their ternary representation. Odlyzko and Stanley [5] offered similar closed-form descriptions of the sequences $S(0, 3^n)$ and $S(0, 2 \cdot 3^n)$, for n any nonnegative integer. Their work also suggested an overarching dichotomy among Stanley sequences, in which the more "well-structured" sequences (such as $S(0)$) follow one asymptotic growth pattern, while more "disorderly" sequences follow another.

Conjecture 1.1 (*Based on Work by Odlyzko and Stanley [5]*). *Let $S(A) = \{a_n\}$ be a Stanley sequence. Then, for all n large enough, one of the following two patterns of growth is satisfied.*

- Type 1. *For some value $\alpha = \alpha(A)$, the following limits exist and are bounded as follows:*

$$\alpha/2 \leq \liminf a_n/n^{\log_2 3} \leq \limsup a_n/n^{\log_2 3} \leq \alpha, \quad \textit{or}$$

- Type 2. $a_n = \Theta(n^2/\log n)$.

Remark. The original paper [5] considered the first type of growth in the case of $\alpha = 1$ only. However, if α is so restricted, the conjecture is certainly false, with $S(0, 1, 7)$ being one counterexample, requiring $\alpha = 10/9$. (This assertion is simple to prove with machinery we present in Sections 2 and 3.)

The closed-form descriptions given in [1] for $S(0, 3^n)$ and $S(0, 2 \cdot 3^n)$ demonstrate that these sequences do indeed follow Type 1 growth, but they are by no means the only such sequences. The justification given in [5] for conjecturing Type 2 growth is a non-constructive probabilistic method that suggests, but does not prove, that a "random" Stanley sequence should follow Type 2 growth. However, no particular sequence has yet been shown to be of Type 2. Gerver [2] has computed the sequence $S(0, 4)$ up to $a_n \approx 2.5 \times 10^6$ and has verified the conjectured growth, also observing interesting fluctuations in the density of the sequence, which empirically occur according to a geometric series. Lindhurst [3] also provides data to support the notion that $S(0, 4)$ follows Type 2 growth. No simple closed form for $S(0, 4)$ is known.

Erdős et al. [1] posed several problems similar to Conjecture 1.1 regarding the density of Stanley sequences. In a recent paper, Moy [4] solved one of these questions by showing that all Stanley sequences $\{a_n\}$ satisfy the asymptotic bound

$$a_n \leq n^2/(2 + \epsilon).$$

Another problem posed in [1] is that of finding a Stanley sequence $\{a_n\}$ for which $\lim_{n\to\infty}(a_n - a_{n-1}) = +\infty$. This remains open; however, Savchev and Chen [8] answered a related question of [1] in the affirmative: there does exist a sequence $\{a_n\}$ for which $\lim_{n\to\infty}(a_n - a_{n-1}) = +\infty$ and such that $\{a_n\}$ is a *maximal 3-free set* — that is, a 3-free set that is not a proper subset of any other 3-free set. Erdős et al. [1] and Moy [4] appear to have assumed that Stanley sequences are maximal 3-free sets, though in Corollary 4.2 we will show that this assumption is false.

In this paper, we approach the conjectured dichotomy among Stanley sequences from the perspective of local structure, rather than asymptotic behavior. We begin by defining the *independent* Stanley sequences and the more general class of *regular* Stanley sequences. In Section 2, we show that all regular Stanley sequences follow Type 1 growth and conjecture that every Stanley sequence that is not regular follows Type 2 growth.

Definition (*See* Example 2.2). We say that a Stanley sequence $S(A) = \{a_n\}$ is *independent* if there exists a constant $\lambda = \lambda(A)$, called the *character*, such that, for all sufficiently large k, the equations

$$a_{2^k+i} = a_{2^k} + a_i, \tag{1}$$

$$a_{2^k} = 2a_{2^k-1} - \lambda + 1 \tag{2}$$

hold whenever $0 \leq i < 2^k$. We say that an integer k_0 is *adequate* if (i) these equations are satisfied for all $k \geq k_0$ and (ii) the minimal set A generating $S(A)$ does not contain the element $a_{2^{k_0}}$.

Definition (*See* Example 2.5). We say that a Stanley sequence $S(A) = \{a_n\}$ is *regular* if there exist constants λ, σ and an independent Stanley sequence $\{a'_n\}$, having character λ, such that: