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Permutations sortable by deques and by two stacks in parallel

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ABSTRACT

Recently Albert and Bousquet-Mélou (2015) obtained the solution to the long-standing problem of the number of permutations sortable by two stacks in parallel (tsip). Their solution was expressed in terms of functional equations. We show that the equally long-standing problem of the number of permutations sortable by a double-ended queue (deque) can be simply related to the solution of the same functional equations. Subject to plausible, but unproved, conditions, the radius of convergence of both generating functions is the same. Numerical work confirms this conjecture to 15 significant digits. Further numerical work suggests that the coefficients of the deque generating function behave as $\kappa_d \cdot \mu^n \cdot n^{-3/2}$, where $\mu = 8.28140220763657\ldots$, while the coefficients of the corresponding tsip generating function behave as $\kappa_p \cdot \mu^n \cdot n^{\gamma}$ with $\gamma \approx -2.47326$. The constants κ_d and κ_p are also estimated.

Inter alia, we study the asymptotics of quarter-plane loops, starting and ending at the origin, with weight a given to north-west and east-south turns. The critical point varies continuously with a, while the corresponding exponent variation is found to be continuous and monotonic for $a > -1/2$, but discontinuous at $a = -1/2$.

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1. Introduction

The problem of *pattern-avoiding permutations* appears to have been first considered by MacMahon [10]. However Knuth [9] was the first to consider a number of classic data structures from the point of view of the permutations they could produce from the identity permutation (or, equinumerously, which permutations could produce the identity permutation). For the data structures considered – stacks and input-restricted deques – Knuth showed that of the $n!$ possible input permutations of length n only $C_n \sim const \cdot 4^n \cdot n^{-3/2}$ and $S_n \sim const \cdot (3+2\sqrt{2})^n \cdot n^{-3/2}$ could be sorted by stacks and input-restricted deques, respectively. This is a consequence of the fact that only 231-avoiding permutations are stack-sortable, as we discuss below.

Shortly thereafter, a number of authors, notably Even and Itai [7], Pratt [11] and Tarjan [12] considered more general data structures. Foremost among these were two stacks in parallel, two stacks in series and deques. A deque, illustrated in Fig. 2, is a double-ended queue, with insertions and deletions allowed at either end. Until recently [2], none of these had been solved. As remarked above, simple stacks can sort any permutation that does not contain three successive (but not necessarily consecutive) elements in the order 231. We write this as the class $Av(231)$, that is, the class of 231-avoiding permutations. For example 1 573 642 cannot be sorted by a stack as the elements 562 (among other sub-sequences) are in the forbidden relative order. Input-restricted deques are describable by the class $Av(4231,\ 3241)$, whereas Pratt [11] showed that the analogous description for deques (without input or output restrictions) requires infinitely many minimal forbidden patterns.

To establish a notation, let p_n denote the number of permutations of length n that can be produced by two parallel stacks, let d_n be the corresponding quantity for deques, and let s_n be the corresponding quantity for two stacks in series. We name the corresponding generating functions

$$P(t) = \sum p_n t^n, \qquad D(t) = \sum d_n t^n, \quad \text{and} \quad S(t) = \sum s_n t^n.$$

In 2010, Albert, Atkinson and Linton [1] studied these problems with a view to establishing upper- and lower-bounds to the relevant growth constants. For deques they found $7.890 < \mu_d < 8.352$ and for tsips they found $7.535 < \mu_p < 8.3461$, and commented that the actual growth constants may be equal, and may possibly be equal to exactly 8. As we show, the two growth constants do indeed appear to be equal, but to a slightly higher value, 8.28140....

In 2015 Albert and Bousquet-Mélou [2] found two coupled functional equations that give the generating function $P(t)$. Unfortunately their representation does not allow for a single equation for the generating function, nor does it allow the asymptotics of p_n to be obtained. However it does offer, in principle, a polynomial-time algorithm to obtain the coefficients p_n. Other unanswered questions include the nature of the solution. Is it D-finite, differentially algebraic, or neither? The answer to these questions is not known. However, we can establish that 1337 terms of the series expansion is not enough to find either a D-finite or differentially algebraic solution.

In their solution, Albert and Bousquet-Mélou first encoded the operations involved in sorting a permutation as words over the alphabet $I_1,\ I_2,\ O_1,\ O_2$, representing the input to, or output from, stack number 1 or stack number 2 respectively. Any sorting of a permutation can be effected by an operation sequence comprising a word in this alphabet, subject to certain constraints. They then pointed out that such words could be considered from two other points of view. The first is a mapping to quarter-plane walks that return to the origin. More precisely, these are lattice walks in $\mathbb{N} \times \mathbb{N}$ that start and end at the origin, making the identification $I_1 \equiv N,\ I_2 \equiv E,\ O_1 \equiv S,\ O_2 \equiv W$, where $N,\ E,\ S,\ W$, denote steps to the north, east, south and west respectively. Then operation sequences on words map to loops that return to the origin. The number of such loops of length $2n$ is given by $C_n C_{n+1}$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan number. However this is not a bijection as a given permutation may correspond to more than one loop.

An alternative representation was given in terms of two-coloured arches, in which the operation sequences were encoded as arches drawn between numbered points on a line, with arches above the line being of a different colour than those below the line. For further details of these connections the reader is referred to [2].