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Obstructions for two-vertex alternating embeddings of graphs in surfaces

Bojan Mohar [1], Petr Škoda

Department of Mathematics, Simon Fraser University, 8888 University Drive, Burnaby, BC, Canada

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ABSTRACT

A class of graphs that lies strictly between the classes of graphs of genus (at most) $k-1$ and k is studied. For a fixed orientable surface $\mathbb{S}_k$ of genus k, let $\mathcal{A}_{xy}^k$ be the minor-closed class of graphs with terminals x and y that either embed into $\mathbb{S}_{k-1}$ or admit an embedding Π into $\mathbb{S}_k$ such that there is a Π-face where x and y appear twice in the alternating order. In this paper, the obstructions for the classes $\mathcal{A}_{xy}^k$ are studied. In particular, the complete list of obstructions for $\mathcal{A}_{xy}^1$ is presented.

1. Introduction

For a simple graph G, let $g(G)$ be the *genus* of G, that is, the minimum k such that G embeds into the orientable surface $\mathbb{S}_k$. A *combinatorial embedding* Π of G is a pair (π, λ) where π assigns each vertex $v \in V(G)$ a cyclic permutation of edges adjacent to v called the *local rotation* around v and the function $\lambda : E(G) \to \{-1, 1\}$ describes the signature of edges when Π is non-orientable. A *Π-face* is a walk in G around a face of Π (for a formal definition see for example [11]). Vertices $v_1, \ldots, v_k$ are *Π-cofacial* if there is a Π-face where the vertices $v_1, \ldots, v_k$ appear in some order. The *Euler genus* $\widehat{g}(\Pi)$ of Π is given as the number $2 - |V(G)| + |E(G)| - |F(\Pi)|$ where $F(\Pi)$ is the set of Π-faces. Similarly *Euler genus* $\widehat{g}(G)$ of G is the minimum $\widehat{g}(\Pi)$ of a combinatorial embedding Π of G. Note that $\widehat{g}(G)$ is the minimum k such that either k is even and G embeds into the orientable surface $\mathbb{S}_{k/2}$ or G embeds into the non-orientable surface $\mathbb{N}_k$.

For an edge e of G, the two standard graph operations, *deletion of* e, $G - e$, and *contraction of* e, G/e, are called *minor operations* and are denoted by $G * e$ when no distinction is necessary. A graph H

E-mail address: pskoda@sfu.ca (P. Škoda).

[1] On leave from: IMFM & FMF, Department of Mathematics, University of Ljubljana, Ljubljana, Slovenia.

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is a *minor* of G if H is obtained from a subgraph of G by a sequence of minor operations. A family of graphs $\mathcal{C}$ is *minor-closed* if, for each graph $G \in \mathcal{C}$, all minors of G belong to $\mathcal{C}$. A graph G is a (*minimal*) *obstruction* for a family $\mathcal{C}$ if G does not belong to $\mathcal{C}$ but for every edge e of G, both $G - e$ and G/e belong to $\mathcal{C}$. The well-known result of Robertson and Seymour [14] asserts that the list of obstructions is finite for every minor-closed family of graphs.

For a fixed surface $\mathbb{S}_k$, the graphs that embed into $\mathbb{S}_k$ form a minor-closed family and it is of general interest to understand the sets of obstructions Forb($\mathbb{S}_k$) for these families. Unfortunately, Forb($\mathbb{S}_1$) already contains thousands of graphs and is not yet determined [6]. We approach the problem by studying graphs in Forb($\mathbb{S}_k$) of small connectivity (see [10]).

In this paper we study a phenomenon that arises when joining two graphs by two vertices. Given graphs G_1 and G_2 such that $V(G_1) \cap V(G_2) = \{x, y\}$, the union of G_1 and G_2, that is the graph $(V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$, is the *xy-sum* of G_1 and G_2. The vertices x and y are called *terminals* and will be often distinguished from other vertices throughout this paper. To determine the genus of the xy-sum of G_1 and G_2, it is necessary to know if G_1 and G_2 have a minimum genus embedding Π such that there is a Π-face in which x and y appear twice in the alternating order (see [4,5]). For vertices $x, y \in V(G)$, we say that G is *xy-alternating on* $\mathbb{S}_k$ if $g(G) = k$ and G has an embedding Π of genus k with a Π-face $W = v_1 \dots v_r$ and indices $i_1, \dots, i_4$ such that $1 \le i_1 < i_2 < i_3 < i_4 \le r$, $v_{i_1} = v_{i_3} = x$, and $v_{i_2} = v_{i_4} = y$.

A graph G is *k-connected* if G has at least $k + 1$ vertices and G remains connected after deletion of any $k - 1$ vertices. A graph has *connectivity* k if it is k-connected but not $(k + 1)$-connected. When G has connectivity 1, vertices whose removal render G disconnected are called *cutvertices*. A *block* of G is a maximal subgraph of G that is 2-connected or an edge not contained in any cycle of G. An *endblock* is a block that contains at most one cutvertex.

To determine minimal obstructions of connectivity 2, we need to know which graphs are minimal *not xy*-alternating (see [10]). For $k \ge 1$, let $\mathcal{A}_{xy}^k$ be the class of graphs with terminals x and y that are either embeddable in $\mathbb{S}_{k-1}$ or are xy-alternating on $\mathbb{S}_k$. When performing minor operations on *graphs with terminals*, we do not allow a contraction identifying two terminals to a single vertex. Also, when contracting an edge joining a terminal and a non-terminal vertex, the new vertex is a terminal. Thus the number of terminals of a minor is the same as of the original graph. A homomorphism of two graphs with terminals is an isomorphism if it is a graph isomorphism and (non-)terminals are mapped onto (non-)terminals. Also, the two terminals may be interchanged. Under these restrictions, $\mathcal{A}_{xy}^k$ is a minor-closed family of graphs such that each graph has two terminals. Let $\mathcal{F}_{xy}^k$ be the set of minimal obstructions for $\mathcal{A}_{xy}^k$, that is, a graph G belongs to $\mathcal{F}_{xy}^k$ if $G \notin \mathcal{A}_{xy}^k$ and, for each edge $e \in E(G)$ and each allowed minor operation $*$, $G * e \in \mathcal{A}_{xy}^k$. It is shown in Section 2 that $\mathcal{F}_{xy}^k$ is finite for each $k \ge 1$. Note that each vertex of a graph in $\mathcal{F}_{xy}^k$ has degree at least 3 except possible when it is a terminal.

A *Kuratowski graph* is a graph isomorphic to K_5, the complete graph on five vertices, or to $K_{3,3}$, the complete bipartite graph on a pair of ternary partite sets. For a fixed Kuratowski graph K, a *Kuratowski subgraph* in G is a minimal subgraph of G that contains K as a minor. A *K-graph* L in G is a subdivision of K_4 or $K_{2,3}$ that can be extended to a Kuratowski subgraph in G. We are using extensively the following well-known theorem.

Theorem 1 (*Kuratowski [8]*)**.** *A graph is planar if and only if it does* not *contain a Kuratowski subgraph.*

Let G be a 2-connected graph. Each vertex of degree different from two is a *branch vertex*. A *branch* of G is a path in G whose endvertices are branch vertices and such that each intermediate vertex has degree 2.

Let H be a subgraph of G. An *H-bridge* in G is a subgraph of G which is either an edge not in H but with both ends in H, or a connected component of $G - V(H)$ together with all edges which have one end in this component and the other end in H. For a H-bridge B, the *interior* of B, B°, is the set $E(B) \cup (V(B) \setminus V(H))$ containing the edges of B and the vertices inside B. Thus, $G - B^\circ$ is the graph obtained from G by deleting B.

Let B be an H-bridge in G. The vertices in $V(B) \cap V(H)$ are called *attachments* of B. The bridge B is a *local bridge* if all attachments of B lie on a single branch of H.